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Approximations in space of quaternionic-valued functions $L^2(\mathbb{R}, Q)$ using Hilbert Transform of wavelets

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Abstract. In one-dimensional space of quaternion-valued square-integrable functions $L^2(\mathbb{R}, Q)$, the Hilbert transform of wavelets has been used to approximate functions. It has been shown that wavelets with multiple vanishing moments are effective in approximating smooth functions in one dimensional Quaternion field $L^2(\mathbb{R}, Q)$. Additionally, we also demonstrated that a function's Hölder continuity helps improve the decay of wavelet coefficients, making the approximation more efficient. Finally, We present a result that connects the dyadic scale differential operator with the Hilbert transform of wavelets in the space $L^2(\mathbb{R}, Q)$ over Q quaternions, providing a method to further reduce the wavelet coefficients in this quaternionic setting.

Keywords: Quaternionic Hilbert spaces, Hilbert transform, Fourier transform, wavelets, Quaternions
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1. Introduction

Fourier and wavelet analyses constitute fundamental tools in harmonic analysis and signal processing, offering powerful mechanisms for frequency and time–frequency representations of signals. Traditionally developed in real- and complex-valued settings, these methods have been progressively extended to accommodate broader classes of functions and transforms. Recent work on the Fourier transform under generalized integration frameworks, such as the Henstock–Kurzweil and continuous primitive integrals, highlights the importance of

analytical flexibility when dealing with irregular or non-absolutely integrable signals [16]. These developments reflect a growing need for mathematical frameworks capable of handling increasingly complex data structures.
Integral transforms play a central role in extracting structural information from signals. In particular, the Hilbert transform is a key operator in harmonic analysis, closely linked to the Fourier transform and analytic signal theory. It provides a natural way to encode phase information and has been widely used in modulation analysis, envelope detection, and instantaneous frequency estimation. At the same time, other integral transforms, such as the Radon transform, raise fundamental questions related to uniqueness and reconstruction, as demonstrated in recent studies on uniqueness and non-uniqueness phenomena [13]. Together, these results emphasize the subtle interplay between transform theory, signal structure, and dimensionality. Quaternion-valued analysis offers a coherent framework in which these classical transforms can be unified and extended. By representing multi-component signals as quaternion-valued functions, one can preserve directional and phase relationships that are otherwise obscured in scalar or component-wise approaches. Within this setting, quaternion Fourier transforms naturally generalize classical Fourier analysis, while quaternion-valued Hilbert transforms provide a higher-dimensional analogue of analytic signal construction. Such extensions allow simultaneous treatment of amplitude, phase, and orientation, which is particularly relevant for signals arising from human-centred systems involving motion, perception, and spatial interaction. Wavelet analysis further enriches this framework by enabling localisation in both time and frequency. Developments in generalized wavelet theories, including linear canonical and Bessel-type wavelet transforms, have underscored the importance of continuity and stability in transform-based analysis [2]. Quaternion-valued wavelets build upon these ideas, offering multiresolution representations for quaternion-valued signals while maintaining geometric coherence across scales. When combined with quaternion-valued Hilbert transforms, wavelet analysis provides a powerful tool for studying local phase and directional features in non-stationary, multi-component signals. From a theoretical perspective, the incorporation of the Hilbert transform into quaternion-valued harmonic analysis raises fundamental questions concerning boundedness, commutation with differential operators, admissibility conditions, and uncertainty principles. These issues are closely related to broader developments in Fourier and Radon transform theory, including questions of generalized integration and uniqueness [16, 13]. Addressing such questions contributes not only to the mathematical depth of quaternion-valued analysis but also to the robustness of associated signal processing methodologies.
In application-driven contexts, quaternion-valued Fourier, Hilbert, and wavelet transforms offer clear advantages for modelling human-centred data. They enable a unified analysis of amplitude, phase, orientation, and scale, which is essential in areas such as human motion analysis, biomedical signal processing, medical imaging, and human–machine interaction. By preserving intrinsic structural relationships within data, quaternion-valued harmonic analysis provides a mathematically rigorous and practically relevant framework for analysing complex signals. Overall, the integration of quaternion-valued analysis with Fourier, Hilbert, and wavelet transforms represents a natural evolution of harmonic analysis in response to increasingly structured and multi-dimensional data. Grounded in recent advances in transform theory and motivated by human-based applications, this framework offers a solid foundation for further theoretical investigation and interdisciplinary research.

Function approximation in quaternionic-valued spaces, particularly in $L^2(\mathbb{R}, Q)$, has drawn significant attention due to its applications in quantum physics, image processing, and signal analysis. Quaternions, extending complex numbers into a non-commutative algebra, enable the modeling of more complex multivariable systems. Functions taking values in quaternions exhibit properties that differ from standard real- or complex-valued functions, which directly impacts their approximation and analysis. Perturbation of Wavelets of Quaternionic-Valued Functions and Polynomial approximation of regular functions in a quaternionic variable are given in [1, 15, 18].

Wavelets are powerful tools for approximating such functions because they capture localized variations across multiple scales in both time and frequency domains. The Hilbert transform, a linear operator often used to examine frequency components of signals, provides essential insights into the harmonic structure of quaternionic-valued functions and supports effective approximation strategies. Spline-based approximations became significant in the 1970s, while it is observed in [3, 14] that for a real wavelet ζ, its Hilbert transform $\mathscr{H}\mathrm{f}$ remains a real wavelet with the same energy and admissibility. Hilbert transforms of Gabor and Wilson systems were studied by Jarrah and Panwar [7], with further details in [4, 10]. Walnut [17] connected vanishing moments to the decay of wavelet coefficients, and Holschneider and Tchamitchian [6] showed that uniform continuity is reflected in small-scale wavelet decay. Mallat [11, 12] proved that a wavelet with n vanishing moments can be expressed as the n-th derivative of a function θ, making the wavelet transform a multiscale differential operator. Approximations, Vanishing moments related to Hilbert transform of wavelets are given in [8, 9].

Functions in $L^2(\mathbb{R}, Q)$ can be decomposed using wavelet expansions, offering a multi-scale representation. The Hilbert transform enhances this framework by isolating components with distinct phase characteristics. Together, these tools facilitate robust approximations that account for smoothness, regularity, and decay conditions of the functions. Approximation quality is further improved by leveraging wavelet properties such as vanishing moments, which enhance performance for functions with specified smoothness requirements. Additionally, the decay of wavelet coefficients across scales ensures that approximations remain accurate even as the resolution changes.

This paper focuses on approximating quaternionic-valued functions in $L^2(\mathbb{R}, Q)$, where Q denotes the quaternions. Our study is divided into two parts. First, we show that the wavelet coefficients of a square-integrable function decrease rapidly as $j \to \infty$, with the rate influenced by the smoothness of the function and the number of vanishing moments of the chosen wavelet. We then provide sufficient conditions for this coefficient decay in functions belonging to $L^2(\mathbb{R}, Q)$. Furthermore, we explore the link between the Hilbert transform of wavelets and the dyadic scale differential operator, applying it to approximate functions $\mathrm{f} \in C^n$ with bounded n^{th}-order derivatives.

2. Preliminaries

In this section, we introduce the foundational concepts necessary to understand our approach to approximating quaternionic-valued functions in $L^2(\mathbb{R}, Q)$ using the Hilbert transform of wavelets. We begin with a brief review of quaternions and the corresponding function spaces. Subsequently, we discuss the key analytical tools, including wavelets and the Hilbert transform, which underpin our approximation methodology.

2.1. **Quaternions and Quaternionic-Valued Functions.** A quaternion $q \in Q$ is an element of the quaternion algebra Q, and can be expressed in the canonical form:

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$, and the imaginary units i, j, k satisfy the fundamental relations:

$$i^2 = j^2 = k^2 = ijk = -1.$$

The set of all such quaternions constitutes a four-dimensional division algebra over the real field $\mathbb{R}$, implying that every non-zero quaternion has a unique multiplicative inverse. Unlike real or complex multiplication, quaternionic multiplication is *non-commutative*; that is, for general quaternions p and q, it holds that $pq \neq qp$.

Throughout this paper, Q denotes the quaternion algebra equipped with the standard conjugation $q \mapsto \overline{q}$ and modulus $|q| = \sqrt{q\overline{q}}$. We consider quaternion-valued functions $f : \mathbb{R} \to Q$. The space $L^2(\mathbb{R}, Q)$ consists of all measurable functions for which

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty.$$

Since the quaternion modulus is real-valued and multiplicative, this definition is well posed and yields a normed space.

The quaternionic inner product on $L^2(\mathbb{R}, Q)$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, dx,$$

which is conjugate-linear in the first argument and right-linear in the second. Owing to the noncommutativity of Q, the order of multiplication in the integrand is essential. However, in the present work we only estimate the modulus $|\langle f, g \rangle|$, which is real-valued. Consequently, quaternion noncommutativity does not influence the magnitude estimates derived later.

A function $f : \mathbb{R} \to Q$ is said to be Hölder continuous of order $\beta \in (0, 1)$ if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^{\beta}, \qquad \forall x, y \in \mathbb{R}.$$

This definition relies solely on the quaternion norm and therefore does not involve any commutative structure.

Let $\zeta \in L^2(\mathbb{R}, Q)$ be a quaternion-valued wavelet. For $j, k \in \mathbb{Z}$, its dilated and translated versions are defined by

$$\zeta_{j,k}(x) = 2^{j/2} \zeta(2^j x - k).$$

We assume that ζ satisfies the vanishing moment conditions

$$\int_{\mathbb{R}} \zeta(x) \, dx = 0, \qquad \int_{\mathbb{R}} x \, \zeta(x) \, dx = 0,$$

and that $x\zeta(x) \in L^1(\mathbb{R}, Q) \cap L^2(\mathbb{R}, Q)$. Since the variable x is real-valued, multiplication by x commutes with quaternion-valued functions, and hence the vanishing moment conditions are unaffected by quaternion noncommutativity.

The Hilbert transform $\mathscr{H}$ of a quaternion-valued function $f \in L^2(\mathbb{R}, Q)$ is defined by

$$\mathscr{H} f(x) = \frac{1}{\pi} \, \text{p.v.} \int_{\mathbb{R}} \frac{f(t)}{x - t} \, dt,$$

where the integral is understood componentwise. Equivalently, $\mathscr{H}$ may be defined via the Fourier multiplier $-i\,\mathrm{sgn}(\xi)$, which is real-valued and therefore commutes with quaternion multiplication. As a result, the Hilbert transform is bounded on $L^2(\mathbb{R}, Q)$ and preserves the scale behaviour of wavelets without introducing additional noncommutative effects.

Although the underlying algebra Q is noncommutative, all decay estimates obtained in this work depend only on norm-based regularity assumptions, real-valued scaling and translation, vanishing moment conditions, and absolute values of inner products. Therefore, quaternion noncommutativity does not affect the magnitude of wavelet–Hilbert coefficients, while it remains relevant for phase and directional information.

For a quaternion-valued function $\mathrm{f} : \mathbb{R} \to Q$, the function is said to be *square-integrable* provided that

$$\int_{\mathbb{R}} |\mathrm{f}(\mathrm{x})|^2 \, d\mathrm{x} < \infty,$$

where the quaternionic modulus is defined by

$$|\mathrm{f}(\mathrm{x})| = \sqrt{a^2 + b^2 + c^2 + d^2},$$

for $\mathrm{f}(\mathrm{x}) = a(\mathrm{x}) + b(\mathrm{x})i + c(\mathrm{x})j + d(\mathrm{x})k$. The collection of all such functions forms the space $L^2(\mathbb{R}, Q)$, equipped with the norm

$$\|\mathrm{f}\|_{L^2(\mathbb{R},Q)} = \left(\int_{\mathbb{R}} |\mathrm{f}(\mathrm{x})|^2 \, d\mathrm{x} \right)^{1/2}.$$

2.2. **Wavelets and the Wavelet Transform.** A *wavelet* ζ is a function used to analyze signals or functions across different scales and positions. It is localized in both the spatial and frequency domains. In the quaternionic setting, a wavelet is defined analogously to the real-valued case, but its values lie within the quaternion algebra.

For a function $\mathrm{f} \in L^2(\mathbb{R}, Q)$, the *wavelet transform* at scale j and translation k is given by

$$\langle \mathrm{f}, \zeta_{j,k} \rangle = \int_{\mathbb{R}} \mathrm{f}(\mathrm{x}) \, \overline{\zeta_{j,k}(\mathrm{x})} \, d\mathrm{x},$$

where the family of wavelets $\zeta_{j,k}(\mathrm{x})$ is generated by scaling and shifting a single mother wavelet ζ according to

$$\zeta_{j,k}(\mathrm{x}) = 2^{j/2} \zeta(2^j \mathrm{x} - k).$$

Here, j denotes the scale parameter, k represents the translation, and the factor $2^{j/2}$ ensures energy normalization across scales.

2.3. **Hilbert Transform.** The *Hilbert transform* [5] is an essential operator in harmonic and signal analysis, often used to obtain the analytic representation of a signal.

For quaternion-valued functions, the Hilbert transform is applied independently to each of the four real components. Thus, if

$$(\mathrm{x}) = a(\mathrm{x}) + b(\mathrm{x})i + c(\mathrm{x})j + d(\mathrm{x})k,$$

then its Hilbert transform is defined as

$$\mathscr{H}\mathrm{f}(\mathrm{x}) = \mathscr{H}a(\mathrm{x}) + i\,\mathscr{H}b(\mathrm{x}) + j\,\mathscr{H}c(\mathrm{x}) + k\,\mathscr{H}d(\mathrm{x}),$$

where each component is computed using the standard real Hilbert transform.

Now, we prove a result in the form of lemma which will be used in the main results.

Lemma 2.1. *Let* $\zeta \in L^2(\mathbb{R}, Q)$. *If* $\mathrm{x}^n\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$ *for some* $n \in \mathbb{N}$, *then* $\mathrm{x}^p\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$ *for all* $p = 0, 1, \ldots, n$.

Proof. Since $\zeta(\mathrm{x})$ is quaternion-valued, the term $\mathrm{x}^n\zeta(\mathrm{x})$ represents the product of a real polynomial and a quaternionic function. The modulus of a quaternion is defined as the square root of the sum of the squares of its real components, which ensures that this norm behaves consistently under such multiplication. To confirm the square-integrability of $\mathrm{x}^n\zeta(\mathrm{x})$, we proceed as follows.

Given that $\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$, it follows that

$$\int_{\mathbb{R}} |\zeta(\mathrm{x})|^2 \, d\mathrm{x} < \infty,$$

where the quaternionic modulus is defined by

$$|\zeta(\mathrm{x})|^2 = \zeta(\mathrm{x}) \, \overline{\zeta(\mathrm{x})},$$

and $\overline{\zeta(\mathrm{x})}$ denotes the quaternionic conjugate of $\zeta(\mathrm{x})$.

For any integer $n \in \mathbb{N}$, the modulus of $\mathrm{x}^n\zeta(\mathrm{x})$ satisfies

$$|\mathrm{x}^n\zeta(\mathrm{x})|^2 = (\mathrm{x}^n\zeta(\mathrm{x})) \, \overline{(\mathrm{x}^n\zeta(\mathrm{x}))} = \mathrm{x}^{2n}\zeta(\mathrm{x}) \, \overline{\zeta(\mathrm{x})}.$$

Hence, $\mathrm{x}^n\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$ if and only if

$$\int_{\mathbb{R}} |\mathrm{x}^n\zeta(\mathrm{x})|^2 \, d\mathrm{x} = \int_{\mathbb{R}} \mathrm{x}^{2n}|\zeta(\mathrm{x})|^2 \, d\mathrm{x} < \infty.$$

Moreover, for each $p = 0, 1, \ldots, n$, we have

$$|\mathrm{x}^p\zeta(\mathrm{x})|^2 = \mathrm{x}^{2p}\zeta(\mathrm{x}) \, \overline{\zeta(\mathrm{x})}.$$

Since the integrability of $\mathrm{x}^n\zeta(\mathrm{x})$ implies that of all lower powers, we deduce that

$$\mathrm{x}^p\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q), \quad \text{for } p = 0, 1, \ldots, n.$$

To verify this more precisely, observe that

$$\int_{\mathbb{R}} |\mathrm{x}|^{2p}|\zeta(\mathrm{x})|^2 \, d\mathrm{x} = \int_{\{\mathrm{x}:|\mathrm{x}|\leq 1\}} |\mathrm{x}|^{2p}|\zeta(\mathrm{x})|^2 \, d\mathrm{x} + \int_{\{\mathrm{x}:|\mathrm{x}|>1\}} |\mathrm{x}|^{2p}|\zeta(\mathrm{x})|^2 \, d\mathrm{x},$$

for each $p = 0, 1, \ldots, n$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}} |\mathrm{x}|^{2p}|\zeta(\mathrm{x})|^2 \, d\mathrm{x} &\leq \|\zeta\|_2^2 + \|\mathrm{x}^n\zeta\|_2^2 \\ &< \infty. \end{aligned}$$

Consequently, $\mathrm{x}^p\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$ for all $p = 0, 1, \ldots, n$. □

3. Main Results

The subsequent result establishes that the wavelet coefficients associated with a square-integrable function exhibit rapid decay as the scale parameter $j \to \infty$, with the rate of decay determined by the smoothness of f and the number of vanishing moments of the corresponding wavelet.

Theorem 3.1. *Let $M \in \mathbb{N}$. Assume that $\mathrm{f} \in L^2(\mathbb{R}, Q) \cap C^M(\mathbb{R}, Q)$ and that its M^{th} derivative satisfies $\mathrm{f}^{(M)} \in L^\infty(\mathbb{R}, Q)$.*

Let $\zeta \in L^2(\mathbb{R}, Q)$ be a compactly supported function such that

$$\mathrm{x}^{M-1}\zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q) \quad \textit{and} \quad \int_{\mathbb{R}} \mathrm{x}^m \zeta(\mathrm{x})\, d\mathrm{x} = 0, \quad 0 \le m \le M-2.$$

Then there exists a positive constant K, depending only on M and f, for which the inequality

$$\left|\langle \mathrm{f},\, \mathscr{H}\zeta_{j,k}\rangle\right| \le K\, 2^{-j(M+\frac{1}{2})}, \qquad \forall\, j, k \in \mathbb{Z},$$

holds, where $\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\zeta(2^j\mathrm{x} - k)$ denotes the dyadic wavelet system and $\mathscr{H}\zeta_{j,k}$ represents its Hilbert transform.

Proof. Since the space $L^2(\mathbb{R}, Q)$ comprises quaternion-valued functions $\mathrm{f} : \mathbb{R} \to Q$ for which the squared norm is integrable, we have

$$\int_{\mathbb{R}} \|\mathrm{f}(\mathrm{x})\|_{\mathbb{Q}}^2\, d\mathrm{x} < \infty.$$

Let $\zeta \in L^2(\mathbb{R}, Q)$ be a quaternion-valued function supported on an interval $[0, b]$ with $b > 0$. The scaled and translated family generated by ζ is expressed as

$$\zeta_{j,k}(\mathrm{x}) = 2^{\frac{j}{2}}\zeta(2^j\mathrm{x} - k),$$

which is supported over the interval $\widetilde{J}_{j,k} = [2^{-j}k, 2^{-j}(k+b)]$, having length $|\widetilde{J}_{j,k}| = 2^{-j}b$.

The Hilbert transform $\mathscr{H}$ acts component-wise on quaternion-valued functions. Specifically, for $\mathrm{f}(\mathrm{x}) = a(\mathrm{x}) + b(\mathrm{x})i + c(\mathrm{x})j + d(\mathrm{x})k$,

$$\mathscr{H}\mathrm{f}(\mathrm{x}) = \mathscr{H}a(\mathrm{x}) + i\,\mathscr{H}b(\mathrm{x}) + j\,\mathscr{H}c(\mathrm{x}) + k\,\mathscr{H}d(\mathrm{x}).$$

Accordingly, for the wavelet system, the Hilbert transform of $\zeta_{j,k}(\mathrm{x})$ can be written as

$$\mathscr{H}(\zeta_{j,k})(\mathrm{x}) = 2^{\frac{j}{2}}\mathscr{H}(\zeta(2^j\mathrm{x} - k)),$$

where the operator $\mathscr{H}$ is applied separately to each quaternionic component of $\zeta(\mathrm{x})$.

Moreover,

$$\int_{\mathbb{R}} \mathrm{x}^p \mathscr{H}\zeta(\mathrm{x})\, d\mathrm{x} = \int_{\mathbb{R}} (\mathscr{H}(a(\mathrm{x})) + i\,\mathscr{H}(b(\mathrm{x})) + j\,\mathscr{H}(c(\mathrm{x})) + k\,\mathscr{H}(d(\mathrm{x})))\, d\mathrm{x},$$

and, for $p = 0, 1, \ldots, M-1$,

$$\int_{\mathbb{R}} \mathrm{x}^p \mathscr{H}(a(\mathrm{x}))\, d\mathrm{x} = \int_{\mathbb{R}} \mathrm{x}^p \mathscr{H}(b(\mathrm{x}))\, d\mathrm{x} = \cdots = 0.$$

Hence, for any quaternionic polynomial $p(\mathrm{x}) = p_0(\mathrm{x}) + p_1(\mathrm{x})i + p_2(\mathrm{x})j + p_3(\mathrm{x})k$ of degree at most $M-1$,

$$\int_{\mathbb{R}} p(\mathrm{x})\, \overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x} = 0.$$

Consider a smooth function $\mathrm{f}(\mathrm{x}) \in C^M(\mathbb{R}, Q)$. Its Taylor expansion about the point $\widetilde{\mathrm{x}}_{j,k}$ is

$$\mathrm{f}(\mathrm{x}) = \sum_{r=0}^{M-1} \frac{(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^r}{r!}\mathrm{f}^{(r)}(\widetilde{\mathrm{x}}_{j,k}) + R_M(\mathrm{x}),$$

where $R_M(\mathrm{x}) = \dfrac{1}{M!}(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^M \mathrm{f}^{(M)}(\epsilon)$, with ϵ lying between $\widetilde{\mathrm{x}}_{j,k}$ and x.

Applying the Hilbert transform to the above expansion yields

$$\mathscr{H}\mathrm{f}(\mathrm{x}) = \sum_{r=0}^{M-1} \frac{1}{r!}\, \mathscr{H}[(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^r]\, \mathrm{f}^{(r)}(\widetilde{\mathrm{x}}_{j,k}) + \mathscr{H}R_M(\mathrm{x}), \tag{3.1}$$

where $\mathscr{H}$ operates on each term component-wise.

Furthermore,

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = \int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\, \overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x} = -\int_{\mathbb{R}} \mathscr{H}\mathrm{f}(\mathrm{x})\, \overline{\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x}. \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = -\sum_{r=0}^{M-1} \frac{1}{r!}\mathrm{f}^{(r)}(\widetilde{\mathrm{x}}_{j,k}) \int_{\mathbb{R}} \mathscr{H}[(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^r]\, \overline{\zeta_{j,k}(\mathrm{x})} d\mathrm{x} - \int_{\mathbb{R}} \mathscr{H}R_M(\mathrm{x})\, \overline{\zeta_{j,k}(\mathrm{x})} d\mathrm{x}. \tag{3.3}$$

Since

$$\int_{\mathbb{R}} \mathscr{H}[(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^r]\, \overline{\zeta_{j,k}(\mathrm{x})} d\mathrm{x} = -\int_{\mathbb{R}} (\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^r\, \overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})} d\mathrm{x} = 0, \quad r = 0, \dots, M-1,$$

it follows that

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| = \left| \int_{\mathbb{R}} \mathscr{H}R_M(\mathrm{x})\, \overline{\zeta_{j,k}(\mathrm{x})} d\mathrm{x} \right|.$$

Using the expression of $R_M(\mathrm{x})$ and applying the Cauchy–Schwarz inequality gives

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| \le \frac{C'}{M!}\, \|\mathrm{f}^{(M)}\|_\infty\, 2^{-M}\, b^{M+\frac{1}{2}}\, 2^{-j(M+\frac{1}{2})},$$

where $C' > 0$ is a constant depending only on ζ. Consequently, the wavelet coefficients exhibit asymptotic decay of order

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| \le K\, 2^{-j(M+\frac{1}{2})}, \qquad j \to +\infty,$$

with $K = \frac{C'}{M!}\|\mathrm{f}^{(M)}\|_\infty\, 2^{-M} b^{M+\frac{1}{2}}$.

□

We now present an example to demonstrate the preceding result.

Example 3.2. Wavelets form an orthonormal set of functions that allow decomposition of signals or functions across multiple scales, providing a framework for multiresolution analysis. Consider the Daubechies wavelet ζ with N vanishing moments. The following characteristics are noteworthy:

- **Compact Support**: ζ is nonzero only over a finite interval, specifically $\left[\frac{-N-1}{2}, \frac{N+1}{2}\right]$.
- **Associated Scaling Function** ϕ: The corresponding scaling function ϕ is supported on $[0, N]$.
- **Vanishing Moments**: ζ satisfies

$$\int_{\mathbb{R}} \mathrm{x}^n \zeta(\mathrm{x})\, d\mathrm{x} = 0, \quad n = 0, 1, \dots, N-1,$$

  ensuring good localization and effective capture of details at various scales.

Let $\mathrm{f} \in L^2(\mathbb{R}, Q)$ be a quaternion-valued function that is at least twice continuously differentiable, with its second derivative bounded, i.e., $\mathrm{f} \in C^2(\mathbb{R}, Q)$ and $\mathrm{f}^{(2)} \in L^\infty(\mathbb{R}, Q)$. We are interested in the wavelet coefficients of the Hilbert-transformed wavelet $\mathscr{H}\zeta_{j,k}$, defined through the inner product

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k} \rangle = \int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\, \overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x},$$

where $\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\zeta(2^j\mathrm{x} - k)$ represents the dyadic scaling and translation of the base wavelet ζ.

Since f is smooth, we can locally approximate it around the center $\widetilde{\mathrm{x}}_{j,k}$ of the wavelet's support using the second-order Taylor expansion:

$$\mathrm{f}(\mathrm{x}) = \mathrm{f}(\widetilde{\mathrm{x}}_{j,k}) + (\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})\mathrm{f}^{(1)}(\widetilde{\mathrm{x}}_{j,k}) + \frac{(\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^2}{2}\mathrm{f}^{(2)}(\epsilon),$$

where ϵ lies between $\widetilde{\mathrm{x}}_{j,k}$ and x. Substituting this into the inner product and noting the vanishing moments of ζ, the contributions from the first two terms vanish, leaving

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k} \rangle = \frac{\mathrm{f}^{(2)}(\epsilon)}{2} \int_{\mathbb{R}} (\mathrm{x} - \widetilde{\mathrm{x}}_{j,k})^2\, \mathscr{H}\zeta_{j,k}(\mathrm{x})\, d\mathrm{x}.$$

Exploiting the relation between the Hilbert transform and the Fourier transform, the integral can be analyzed in the frequency domain. Moreover, because x remains close to $\widetilde{\mathrm{x}}_{j,k}$ within the wavelet's compact support, we can bound $|\mathrm{x} - \widetilde{\mathrm{x}}_{j,k}| \leq C\, 2^{-j}$ for some constant C. The compact support of ζ further ensures the integral is finite. Consequently, the wavelet coefficients satisfy

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k} \rangle| \leq C'\, 2^{-5j/2},$$

where C' depends on the supremum of $|\mathrm{f}^{(2)}(\mathrm{x})|$ and the size of the wavelet's support.

This result illustrates that for sufficiently smooth quaternion-valued signals, the Hilbert-transformed wavelet coefficients decay rapidly with increasing scale j. The rate of decay is governed by two main factors:

- The smoothness of f, through the boundedness of higher-order derivatives.
- The number of vanishing moments of the wavelet ζ, which determines how lower-order polynomial contributions vanish in the inner product.

Thus, both the signal's regularity and the wavelet's structural properties dictate the exponential decay of coefficients.

The following result employs uniform Hölder continuity of a function to derive necessary criteria for the reduction of its wavelet coefficients.

Theorem 3.3. *Let* $\mathrm{f} \in L^2(\mathbb{R}, Q)$ *be Hölder continuous with exponent* $\beta \in (0, 1)$*. Assume* $\zeta \in L^2(\mathbb{R}, Q)$ *is a wavelet satisfying:*

(1) $\mathrm{x}\zeta(\mathrm{x}) \in L^1(\mathbb{R}, Q) \cap L^2(\mathbb{R}, Q)$,

(2) $\int_{\mathbb{R}} \mathrm{x}^p \zeta(\mathrm{x})\, d\mathrm{x} = 0$ *for* $p = 0, 1$.

Then a constant $C > 0$ *exists such that*

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k} \rangle\right| \leq C\, 2^{-j(\beta + \frac{1}{2})},$$

for all $j, k \in \mathbb{Z}$. Here, $\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\zeta(2^j\mathrm{x} - k)$ denotes the wavelet scaled and translated by j and k, and $\mathscr{H}\zeta_{j,k}$ is its Hilbert transform.

Proof. Since we are working in the space $L^2(\mathbb{R}, Q)$, the Hilbert space of square-integrable quaternion-valued functions. The inner product in this space is defined as:

$$\langle \mathrm{f}, g\rangle_Q = \int_{\mathbb{R}} f(\mathrm{x})\overline{g(\mathrm{x})}\, d\mathrm{x},$$

where $\overline{g(\mathrm{x})}$ is the quaternionic conjugate of $g(\mathrm{x})$.

We need to evaluate the quantity:

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle_Q| = \left|\int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x}\right|.$$

Substitute the expression for $\mathscr{H}\zeta_{j,k}$:

$$\mathscr{H}\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\zeta(2^j\mathrm{x} - k),$$

we have

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle_Q = 2^{j/2}\int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\overline{\zeta(2^j\mathrm{x} - k)}\, d\mathrm{x}.$$

Since f is Hölder continuous with exponent β, we know that for any two points x and y in $\mathbb{R}$,

$$|\mathrm{f}(\mathrm{x}) - \mathrm{f}(y)| \le C|\mathrm{x} - y|^\beta.$$

This property will help us bound the integrand in the next steps. Now to simplify the integral we can use change of variables as

$$u = 2^j\mathrm{x} - k \quad \Rightarrow \quad \mathrm{x} = \frac{u+k}{2^j}, \quad d\mathrm{x} = \frac{du}{2^j}.$$

This transforms the inner product to:

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle_Q = 2^{j/2}\int_{\mathbb{R}} \mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}\frac{du}{2^j}.$$

Simplifying the constant factors:

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle_Q = 2^{-j/2}\int_{\mathbb{R}} \mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}\, du.$$

Now we need to estimate the integral:

$$\int_{\mathbb{R}} \mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}\, du.$$

Since ζ is a wavelet with zero mean and first moment, and it is assumed to decay rapidly, we know that $\zeta(u)$ behaves like a rapidly decaying function at large $|u|$. We can focus on the region around $u = 0$, where $\zeta(u)$ has significant contributions.

Let's use the Hölder continuity of f. We can bound $\mathrm{f}\left(\frac{u+k}{2^j}\right)$ by:

$$|\mathrm{f}\left(\frac{u+k}{2^j}\right) - \mathrm{f}\left(\frac{k}{2^j}\right)| \le C\left|\frac{u}{2^j}\right|^\beta = C2^{-j\beta}|u|^\beta.$$

Thus, the integrand is dominated by a term that behaves like:

$$|\mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}| \le C2^{-j\beta}|u|^\beta|\zeta(u)|.$$

Since $\zeta(u)$ decays rapidly, we can bound the integral as follows:

$$\int_{\mathbb{R}} |\mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}|\,du \leq C2^{-j\beta}\int_{\mathbb{R}}|u|^\beta|\zeta(u)|\,du.$$

Let $C' = \int_{\mathbb{R}}|u|^\beta|\zeta(u)|\,du$, which is finite due to the decay of ζ. Hence, the integral can be bounded as:

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle_Q| \leq C2^{-j(\beta+\frac{1}{2})},$$

where the factor $2^{-j\frac{1}{2}}$ comes from the scaling of the wavelet, and C is a constant depending on f, ζ, and the decay rates. □

Example 3.4 (Worked example illustrating coefficient decay)**.** We illustrate the decay estimate

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle\right| \lesssim 2^{-j(\beta+\frac{1}{2})}$$

with a concrete example.

Choice of the signal. Let

$$\mathrm{f}(x) = |x|^\beta, \qquad 0<\beta<1.$$

Then $\mathrm{f} \in L^2(\mathbb{R},Q)$ and f is Hölder continuous of order β, since there exists a constant $C>0$ such that

$$|\mathrm{f}(x)-\mathrm{f}(y)| \leq C|x-y|^\beta, \qquad \forall x,y\in\mathbb{R}.$$

This function has a singularity at the origin and serves as a standard test function for examining wavelet coefficient decay.

Choice of the wavelet. Let $\zeta \in L^2(\mathbb{R},Q)$ be a quaternion-valued wavelet satisfying

$$\int_{\mathbb{R}}\zeta(x)\,dx = 0, \qquad \int_{\mathbb{R}} x\,\zeta(x)\,dx = 0,$$

and

$$x\zeta(x) \in L^1(\mathbb{R},Q)\cap L^2(\mathbb{R},Q).$$

Such assumptions are fulfilled, for instance, by compactly supported quaternion-valued wavelets with two vanishing moments.

Estimation of the coefficient. The wavelet–Hilbert coefficient is given by

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = 2^{-j/2}\int_{\mathbb{R}}\mathrm{f}\left(\frac{u+k}{2^j}\right)\overline{\zeta(u)}\,du.$$

Using the vanishing moment condition $\int_{\mathbb{R}}\zeta(u)\,du=0$, we write

$$\mathrm{f}\left(\frac{u+k}{2^j}\right) = \mathrm{f}\left(\frac{k}{2^j}\right) + \left(\mathrm{f}\left(\frac{u+k}{2^j}\right)-\mathrm{f}\left(\frac{k}{2^j}\right)\right),$$

where the constant term vanishes after integration.

By the Hölder continuity of f, we obtain

$$\left|\mathrm{f}\left(\frac{u+k}{2^j}\right)-\mathrm{f}\left(\frac{k}{2^j}\right)\right| \leq C\,2^{-j\beta}|u|^\beta.$$

Hence,

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle\right| \leq 2^{-j/2}\,2^{-j\beta}\int_{\mathbb{R}}|u|^\beta|\zeta(u)|\,du.$$

Since $\int_{\mathbb{R}} |u|^\beta |\zeta(u)|\, du < \infty$, it follows that

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle\right| \leq C\, 2^{-j(\beta+\frac{1}{2})},$$

where $C > 0$ is independent of j and k.

Remark 3.5 (Schematic interpretation)**.** The above estimate shows that, on a logarithmic scale, the magnitude of the coefficients $|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle|$ decays linearly with respect to the scale parameter j, with slope $-(\beta + \frac{1}{2})$. This decay rate reflects the Hölder regularity of the signal and is unaffected by the Hilbert transform, which primarily modifies the phase rather than the magnitude of the coefficients.

The following theorem extends the preceding result to a more general framework.

Theorem 3.6. *Let* $\mathrm{f} \in L^2(\mathbb{R}, Q)$ *be a quaternion-valued function that is continuously differentiable up to order* n. *Assume that its* n^{th} *derivative, denoted by* $\mathrm{f}^{(n)}$, *satisfies a Hölder condition with exponent* β, *where* $0 < \beta < 1$.

Consider a wavelet $\zeta \in L^2(\mathbb{R}, Q)$ *satisfying the following properties:*

$$\mathrm{x}^{n+1}\zeta(\mathrm{x}) \in L^1(\mathbb{R}, Q) \cap L^2(\mathbb{R}, Q), \tag{3.4}$$

$$\int_{\mathbb{R}} \mathrm{x}^p \zeta(\mathrm{x})\, d\mathrm{x} = 0, \qquad \text{for } p = 0, 1, 2, \ldots, n+1. \tag{3.5}$$

Then, there exists a positive constant C, *depending only on* f *and* n, *such that*

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle\right| \leq C\, 2^{-j\left(n+\beta+\frac{1}{2}\right)},$$

for all integers j, k. *Here,* $\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\zeta(2^j\mathrm{x} - k)$ *represents the scaled and translated form of the wavelet, and* $\mathscr{H}\zeta_{j,k}$ *denotes its Hilbert transform.*

Proof. The wavelet coefficient corresponding to the function f and the Hilbert-transformed wavelet $\mathscr{H}\zeta_{j,k}$ is defined as

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = \int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\, \overline{\mathscr{H}\zeta_{j,k}(\mathrm{x})}\, d\mathrm{x}.$$

Substituting $\mathscr{H}\zeta_{j,k}(\mathrm{x}) = 2^{j/2}\mathscr{H}\zeta(2^j\mathrm{x} - k)$ gives

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = 2^{j/2} \int_{\mathbb{R}} \mathrm{f}(\mathrm{x})\, \overline{\mathscr{H}\zeta(2^j\mathrm{x} - k)}\, d\mathrm{x}.$$

To simplify, we introduce the change of variables $u = 2^j\mathrm{x} - k$, which yields $\mathrm{x} = 2^{-j}(u + k)$ and $d\mathrm{x} = 2^{-j} du$. Substituting these relations gives

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = 2^{j/2} \int_{\mathbb{R}} \mathrm{f}(2^{-j}(u + k))\, \overline{\mathscr{H}\zeta(u)}\, 2^{-j} du,$$

and therefore,

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = 2^{-j/2} \int_{\mathbb{R}} \mathrm{f}(2^{-j}(u + k))\, \overline{\mathscr{H}\zeta(u)}\, du.$$

Because f is assumed to be n-times continuously differentiable and its n^{th} derivative $\mathrm{f}^{(n)}$ is Hölder continuous with exponent β, it can be locally expanded around $2^{-j}u$ as

$$\mathrm{f}(2^{-j}(u + k)) = \mathrm{f}(2^{-j}u) + \sum_{m=1}^{n} \frac{\mathrm{f}^{(m)}(2^{-j}u)}{m!}(2^{-j}k)^m + R_n(k),$$

where the remainder satisfies $|R_n(k)| = O(|k|^{n+\beta} 2^{-j(n+\beta)})$ for $0 < \beta < 1$. Hence,

$$\mathrm{f}(2^{-j}(u+k)) - \mathrm{f}(2^{-j}u) = O(|k|^{\beta} 2^{-j\beta}).$$

Substituting this back into the integral form gives

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = 2^{-j/2} \int_{\mathbb{R}} \mathrm{f}(2^{-j}u)\, \overline{\mathscr{H}\zeta(u)}\, du + 2^{-j/2} \int_{\mathbb{R}} O(|k|^{\beta} 2^{-j\beta})\, \overline{\mathscr{H}\zeta(u)}\, du.$$

The first integral remains bounded since $\mathrm{f} \in L^2(\mathbb{R}, Q)$ and $\mathscr{H}\zeta$ decays rapidly:

$$\left| 2^{-j/2} \int_{\mathbb{R}} \mathrm{f}(2^{-j}u)\, \overline{\mathscr{H}\zeta(u)}\, du \right| \leq C_1.$$

The second integral, involving the error term, can be bounded using the vanishing moment property of ζ, giving

$$\left| 2^{-j/2} \int_{\mathbb{R}} O(|k|^{\beta} 2^{-j\beta})\, \overline{\mathscr{H}\zeta(u)}\, du \right| \leq C_2\, 2^{-j(n+\beta+\frac{1}{2})}.$$

Combining both estimates yields

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| \leq C_1 + C_2 2^{-j(n+\beta+\frac{1}{2})}.$$

For sufficiently large j, the dominant contribution arises from the second term, resulting in

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| \leq C\, 2^{-j(n+\beta+\frac{1}{2})},$$

where $C > 0$ is independent of j and k. □

Recall that a function $\mathrm{f} \in C^n(\mathbb{R})$ that is bounded is said to exhibit a *decay of order* $m \in \mathbb{N}$ if there exists a positive constant C_m such that, for all $\mathrm{x} \in \mathbb{R}$ and each p satisfying $0 \leq p \leq n$, one has

$$|\mathrm{f}^{(p)}(\mathrm{x})| \leq \frac{C_m}{1 + |\mathrm{x}|^m}.$$

Further explanation of this concept can be found in [11].

The next result establishes a connection between the Hilbert transform of wavelets and the dyadic-scale differential operator. This relation allows one to control the decay of the wavelet coefficients $\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle$ and provides a method to approximate functions $\mathrm{f} \in C^n$ whose n^{th} derivative is bounded.

Theorem 3.7. *Let $m, n \in \mathbb{Z}$ with $m \geq n + 2$, and assume ζ is a wavelet satisfying $\zeta, \hat{\zeta} \in L^2(\mathbb{R}, Q)$ along with the following properties:*

(i) $\mathrm{x}^n \zeta(\mathrm{x}) \in L^2(\mathbb{R}, Q)$,
(ii) ζ *has* $(n-1)$ *vanishing moments,*
(iii) the decay estimate $|\zeta(t)| \leq \dfrac{C_m}{1 + |t|^m}$ *holds, and* $\mathscr{H}\zeta$ *decays at the same rate* m.

Under these assumptions, there exists a bounded function μ whose Hilbert transform $\mathscr{H}\mu$ is also bounded and exhibits decay of order m, such that

$$\mathscr{H}\zeta(t) = (-i)^n\, \mathscr{H}\mu^{(n)}(t). \tag{3.6}$$

Moreover, if $\mathrm{f} \in C^n$ *with its* n^{th} *derivative bounded, then the corresponding wavelet coefficients satisfy*

$$\left|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle\right| = O\left(2^{\frac{j(n+1)}{2}}\right).$$

Proof. Let $\zeta \in L^2(\mathbb{R}, Q)$ be a wavelet with $(n-1)$ vanishing moments and decay rate $m \geq n+2$:

$$|\zeta(x)| \leq \frac{C_m}{1+|x|^m}.$$

Its Fourier transform then satisfies

$$\hat{\zeta}(\epsilon) \sim \epsilon^{-n} \quad \text{as} \quad \epsilon \to \infty,$$

indicating that ζ is smooth at high frequencies. The spatial decay ensures that $\hat{\zeta}(\epsilon)$ itself decays sufficiently fast for large ϵ.

Define an auxiliary function μ in the frequency domain by

$$\hat{\mu}(\epsilon) = \frac{\hat{\zeta}(\epsilon)}{\epsilon^n}.$$

This choice guarantees faster decay at high frequencies:

$$\hat{\mu}(\epsilon) \sim \epsilon^{-2n} \quad \text{as} \quad \epsilon \to \infty,$$

so μ possesses sufficient temporal decay. Its Hilbert transform $\mathscr{H}\mu$ is bounded and decays at rate m, satisfying

$$\mathscr{H}\zeta(t) = (-i)^n \frac{d^n}{dt^n}\mathscr{H}\mu(t).$$

Consider the Hilbert-transformed wavelet at scale 2^j and shift k:

$$\mathscr{H}\zeta_{j,k}(t) = 2^{j/2}\mathscr{H}\zeta(2^j t - k),$$

with Fourier transform

$$\hat{\mathscr{H}\zeta}_{j,k}(\epsilon) = 2^{j/2}e^{-2\pi i k\epsilon}\hat{\zeta}(2^{-j}\epsilon).$$

Time differentiation corresponds to multiplication by $(-i\epsilon)^n$ in the frequency domain, ensuring that

$$\mathscr{H}\zeta(t) = (-i)^n \frac{d^n}{dt^n}\mathscr{H}\mu(t).$$

Using the decay of ζ, it follows that

$$|\mathscr{H}\mu^{(p)}(t)| \leq \frac{C_m}{1+|t|^m}, \quad 0 \leq p \leq n,$$

so $\mathscr{H}\mu$ and its derivatives decay sufficiently rapidly at infinity.

For a function $\mathrm{f} \in C^n$ with bounded n-th derivative, the wavelet coefficients are given by

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = \int_{\mathbb{R}} \mathrm{f}(t)\overline{\mathscr{H}\zeta_{j,k}(t)}\,dt.$$

Expressed as a convolution, this becomes

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = (\mathrm{f} * \mathscr{H}\zeta'_{2^j})(2^j k),$$

where

$$\begin{aligned}\mathscr{H}\zeta'_{2^j}(t) &= 2^{-j/2}\mathscr{H}\zeta(-2^{-j}t),\\ \mathscr{H}\zeta'_{2^j}(u) &= i^n 2^{jn}\frac{d^n}{du^n}\mathscr{H}\mu'_{2^j}(u),\\ \mathscr{H}\mu'_{2^j}(u) &= 2^{-j/2}\mathscr{H}\mu(-2^{-j}u).\end{aligned}$$

Thus, we obtain

$$\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle = i^n 2^{jn}\frac{d^n}{du^n}\big(\mathrm{f} * \mathscr{H}\mu'_{2^j}(u)\big),$$

showing explicitly how the n-th derivative of the smoothed function governs the wavelet coefficients.

Finally, for large scales j, it follows that

$$|\langle \mathrm{f}, \mathscr{H}\zeta_{j,k}\rangle| = O(2^{j(n+1/2)}),$$

demonstrating that the contribution of high-scale components diminishes rapidly, which validates the multiscale approximation and ensures control over the approximation error. □

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