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Schauder frames satisfying certain properties

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Abstract. In this article, we define property $\mathscr{U}$ and discuss its existence via examples and counter examples. Also, we prove that a separable Banach space that has a subspace whose dual space is non separable and weakly complete does not have a Schauder frame satisfying property $\mathscr{U}$. Moreover, we investigate Banach spaces having property $(\mathscr{P})$ and discuss the existence of Schauder frames satisfying property $(\mathscr{U})$. Finally, we observe that $L^1[0,1]$ has no Schauder frame satisfying property $\mathscr{U}$.

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1. INTRODUCTION AND PRELIMINARIES

In Hilbert spaces, frames provide stable, possibly redundant, representations of elements. That is, every vector can be written as a (possibly infinite) linear combination of frame elements with coefficients that are bounded and computable.

While working with families of exponentials $\{e^{i\lambda_n t}\}_{n\in\mathbb{Z}}$, Duffin and Schaeffer [4], tried to determine when they are complete or form a Reisz basis for $L^2[a,b]$ etc, and in the process defined frames. They gave the following definition of frame:

A sequence $\{x_n\}$ in a Hilbert space H (with inner product $\langle \cdot, \cdot \rangle$) is said to be a frame (Hilbert) for H, if there exist positive constants A and B such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \quad \text{for all } \ x \in H \tag{1.1}$$

The positive constants A and B, respectively, are called *lower* and *upper frame bound* for the frame $\{x_n\}$ and collectively known as *frame bounds* for the frame $\{x_n\}$. But they are not unique. In fact, any pair of positive constants A and B for which the inequality (1.1), called *frame inequality* for the frame $\{x_n\}$, hold good are called frame bounds for the frame $\{x_n\}$. The supremum of all lower frame bounds and the infimum of all upper frame bounds (of the frame $\{x_n\}$) are called *optimal bounds* or *best bounds* for the frame $\{x_n\}$.

In Banach spaces, the absence of an inner product complicates the direct use of Hilbert-space frames, and this motivates the development of analogous concepts that work in Banach spaces like the notion of Banach frames and Schauder frames. Banach spaces was given by Gröchenig [8]. He gave the following definition of Banach frame for Banach spaces.

Let $\mathcal{B}$ be a non-trivial Banach space and let $\mathcal{B}_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathbb{N}$. Let $\{f_n\}$ be a sequence in $\mathcal{B}^*$ and let $S : \mathcal{B}_d \to \mathcal{B}$ be an operator. Then the pair $(\{f_n\}, S)$ is called a *Banach frame* for $\mathcal{B}$ with respect to $\mathcal{B}_d$, if

(1) $\{f_n(x)\} \in \mathcal{B}_d$, $x \in \mathcal{B}$,

(2) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_{\mathcal{B}} \leq \|\{f_n(x)\}\|_{\mathcal{B}_d} \leq B\|x\|_{\mathcal{B}}, \quad x \in \mathcal{B}, \tag{1.2}$$

(3) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in \mathcal{B}.$$

As in case of frame for a Hilbert space the positive constants A and B, respectively, are called *lower* and *upper* frame bound for the Banach frame $(\{f_n\}, S)$ and collectively known as *frame bounds* for $(\{f_n\}, S)$. The operator $S : \mathcal{B}_d \to \mathcal{B}$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (1.2) is called the *frame inequality* for the Banach frame $(\{f_n\}, S)$. Banach frames were studied in [5, 6, 7].

Schauder frame is a generalization of both Schauder bases and Hilbert frames to the broader setting of Banach spaces. Schauder frames are particularly useful in functional analysis and applied mathematics for representing elements in infinite-dimensional Banach spaces in a stable and redundant way.They were first introduced by Casazza, Han and Larsan [1] and were further studied by Kaushik et al.[11]. More recent contributions on frame theory in Banach spaces can be found in [8, 13].Keeping applications in various other related areas of applied mathematics, Schauder frames were also studied in [2, 3, 9, 10, 12]. The formal definition of Schauder frame is as follows:

Let $\mathcal{B}$ be a Banach space and let $\{x_n\}$ be a sequence in $\mathcal{B}$ and $\{f_n\}$ be sequence in $\mathcal{B}^*$. Then the pair $(\{x_n\}, \{f_n\})$ is called a Schauder frame for $\mathcal{B}$ if

$$x = \sum_{n=1}^{\infty} f_n(x) x_n, \text{ for all } x \in \mathcal{B}$$

Throughout H (with inner product $\langle\cdot,\cdot\rangle$) will denote a separable Hilbert space, $\mathcal{B}$ a Banach space , and $\mathcal{B}^*$ denote the conjugate space of $\mathcal{B}$.

The following result stated in the form of Lemma will be referred in the subsequent work.

Lemma 1.1. [11]. *Let* $(\{f_n\},S)$ *(where* $\{f_n\}\subset\mathcal{B}^*$ *and* $S:\mathcal{B}_d\to E$*) be a Banach frame for* $\mathcal{B}$ *with respect to* $\mathcal{B}_d$*. Then* $(\{f_n\},S)$ *is exact if and only if* $f_n\notin\widetilde{[f_i]}_{i\neq n}$ *for all* $n\in\mathbb{N}$*.*

1.1. **Outline of the work.** In this work, we introduce the concept of property $\mathscr{U}$ and explore its presence through various examples and non-examples. We demonstrate that a separable Banach space which contains a subspace whose dual is nonseparable and weakly complete cannot have a Schauder frame that satisfies property $\mathscr{U}$. Additionally, we study Banach spaces exhibiting property $(\mathscr{P})$ and consider the conditions under which Schauder frames with property $\mathscr{U}$ may exist. As a notable case, we establish that the space $L^1[0,1]$ does not admit a Schauder frame with property $\mathscr{U}$.

2. Main Results

This section opens with the definition of Schauder frames that fulfill property $\mathscr{U}$, followed by an exploration of their existence using both examples and counterexamples. We begin with the following definition:

Definition 2.1. Let $\mathcal{B}$ be a separable Banach space. Let $\{a_n\}\subseteq\mathcal{B}$ and $\{a_n^*\}\subset\mathcal{B}^*$. Then, the pair $(\{a_n\},\{a_n\}^*)$ is said to satisfy property $\mathscr{U}$ if for every $x\in\mathcal{B}$ the series

$$\sum_{n=1}^{\infty}a_n^*(x)a_n$$

converges to x unconditionally.

Let $\{\phi_n\}$ be the unit vector sequence in $\mathcal{B}$ and $\{\phi_n^*\}$ be the unit vectors sequence in $\mathcal{B}^*$. If we define a sequence $\{a_n\}$ in $\mathcal{B}$ by:
$a_1=a_2=\phi_1$ and $a_n=\phi_{n-1}$, $\forall\, n\geq 3$ and a sequence $\{a_n^*\}$ in $\mathcal{B}^*$ by $a_1^*=a_2^*=\frac{\phi_1^*}{2}$ and $a_n^*=\phi_{n-1}^*$, $\forall\, n\geq 3$. Then

$$\|\sum_{j=k}^{k+p}a_j^*(x)a_j\|=\sum_{j=k}^{k+p}|\xi_j|,\ \forall\, x=\{\xi_j\}\in\mathcal{B},$$

for each $k>2$, and $p\in\mathbb{N}$..
Since $\sum_{j=1}^{\infty}|\xi_j|$ converges, it follows that the series $\sum_{j=1}^{\infty}a_j^*(x)a_j$ also converges unconditionally.
Further, for $x\in\mathcal{B}$

$$\sum_{j=1}^{\infty}a_j^*(x)a_j=\sum_{j=1}^{\infty}\phi_j^*(x)\phi_j=x.$$

Hence $(\{a_n\},\{a_n^*\})$ is a Schauder frame satisfying property $'u'$.
Now let $\mathcal{B}=c_0$ and $\{a_n\}\subset\mathcal{B}$ be a sequence defined as $a_n=\{1,1,...,1,0,0,0,......\}$, $\forall n\in\mathbb{N}$

and the sequence $\{a_n^*\} \subsetneq \mathcal{B}^*$ by $a_n^* = \{0, 0, ...1, -1, 0, 0, 0, ...\}$, $\forall\, n \in \mathbb{N}$.
Let $x = \{\xi_n\} \in \mathcal{B}$ be any element. Then

$$\begin{aligned}\sum_{j=1}^{n} a_j^*(x) a_j &= (\xi_1 - \xi_2, 0, 0, ...) + (\xi_2 - \xi_3, \xi_2 - \xi_3, 0, 0, ...) \\ &+ \quad ...... + (\xi_n - \xi_{n+1}, \xi_n - \xi_{n+1}, ...\xi_n - \xi_{n+1}, 0, 0, ...)\end{aligned}$$

This implies

$$\sum_{j=1}^{\infty} a_n^*(x) a_n = \lim_{n\to\infty} (\xi_1, \xi_2, ...\xi_n, 0, 0, ...) = x.$$

Thus, $(\{a_n\}, \{a_n^*\})$ is a Schauder frame for $\mathcal{B}$.
Now, if $\{\eta_n\}$ is the sequence of standard unit vectors in $\mathcal{B}$, then for $k, p \in \mathbb{N}$ and for each $x \in \mathcal{B}$, we get

$$\begin{aligned}\sum_{j=k}^{k+p} a_j^*(x) a_j &= \sum_{j=k}^{k+p} a_j^*(x) \sum_{i=1}^{j} \eta_i \\ &= \sum_{j=k}^{k+p} a_j^*(x) \sum_{i=1}^{k} \eta_i + \sum_{j=k}^{k+p} a_j^*(x) \eta_{k+1} + ...a_{k+p}^*(x) \eta_{k+p}\end{aligned}$$

This yields

$$\| \sum_{j=k}^{k+p} a_j^*(x) a_j \| = \sup_{k \le i \le k+p} | \sum_{j=i}^{k+p} g_j^*(x) |, \forall x \in \mathcal{B}.$$

Thus, for each $x \in \mathcal{B}$, the series $\sum_{n=1}^{\infty} a_n^*(x) a_n$ converges in $\mathcal{B}$ iff the series of scalars $\sum_{n=1}^{\infty} g_n(x)$ converges.
However, for $x_0 = (0, \frac{1}{2}, 0, \frac{1}{3}, ...) \in \mathcal{B}$, the series

$$\sum_{n=1}^{\infty} a_n^*(x_0) = -\frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3}.....$$

converges conditionally.
Hence, the Schauder frame $(\{a_n\}, \{a_n^*\})$ does not satisfy property $'u'$.
This counterexample illustrates that even though a pair can form a Schauder frame, it may not satisfy the stronger requirement of unconditional convergence.

Now, we prove that there is no Schauder frame satsifying property $\mathscr{U}$ in a separable Banach space that has a subspace whose conjugate space is non-separable and weakly complete.

Theorem 2.2. *Let $\mathcal{B}$ be a separable Banach space. Let $\mathcal{B}_1$ be a subspace of $\mathcal{B}$ such that $\mathcal{B}_1^*$ is non-separable and weakly complete. Then $\mathcal{B}$ has no Schauder frame satisfying property $\mathscr{U}$.*

Proof. Suppose $(\{a_n\}, \{a_n^*\})$ $(\{a_n\} \subset \mathcal{B}, \{a_n^*\} \subset \mathcal{B}^*)$ is a Schauder frame satisfying property $\mathscr{U}$. Let $b_n^* = a_n^*/\mathcal{B}_1, \forall n \in \mathbb{N}$ and let $b^* \in \mathcal{B}_1^*$ be any arbitray functional and $a^* \in \mathcal{B}^*$ be an extension of b^* to $\mathcal{B}$.
By hypothesis, $\sum_{n=1}^{\infty} a_n^*(x) a_n$ converges unconditionally to $x \in \mathcal{B}$. Therefore, for each a^* in $\mathcal{B}^*$, the series $\sum_{n=1}^{\infty} a^*(a_n) a_n^*$ is $\sigma(\mathcal{B}^*, \mathcal{B})$ unconditionally convergent to a^* and so the series $\sum_{n=1}^{\infty} b^*(a_n) a_n^*$ is $\sigma(\mathcal{B}^*, \mathcal{B})$ unconditionally convergent to b^* in $\mathcal{B}_1^*$. So $\sum_{n=1}^{\infty} b^*(a_n) a_n^*$

is $\sigma(\mathcal{B}^*, \mathcal{B})$- unconditionally Cauchy.
Now, for each $j \in \mathbb{N}$, define

$$k_j(x) = \sum_{i=1}^{k} |b^*(a_i)a_i^*(x)|, x \in \mathcal{B}.$$

Then, for each $x, y \in \mathcal{B}$, we obtain

$$\begin{aligned} |k_j(x+y)| &= |\sum_{i=1}^{j} b^*(a_i)a_i^*(x+y)| \\ &\leq |k_j(x)|+|k_j(y)|. \end{aligned}$$

Also, for any scalar $\alpha \geq 0$ and $x \in \mathcal{B}$, we have

$$|\alpha k_j(x)| = |\alpha \sum_{i=1}^{j} b^*(a_i)a_i^*(x)| = |k_j(\alpha x)|.$$

So, one may conclude that for each $x \in \mathcal{B}$, the set $\{k_j(x) : j \in \mathbb{N}\}$ is bounded. Therefore $\lim_{n\to\infty} k_j(x) = 0$ exists uniformly in $\mathcal{B}$ for $j \in \mathbb{N}$.
Thus, one can find a constant $C > 0$ satisfying

$$\sum_{j=1}^{n} |b^*(a_j)a_j^*(x)| \leq C\|x\|, \; x \in \mathcal{B} \, \text{and} \, n \in \mathbb{N}.$$

Let $\{\beta_n\}$ be any sequence of scalars with $|\beta_n| \leq 1$, $n \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, we compute

$$\sum_{j=1}^{k} |\beta_j b^*(a_j)b_j^*(x)| \leq C\|x\|, \; x \in \mathcal{B}$$

Thus, for $\alpha_i = 0 \,\text{or}\, 1 \,\text{and}\, j = 1, 2, ..., n$, there exists a constant $C' > 0$. such that

$$\|\sum_{j=1}^{k} \alpha_j b^*(a_j)a_j^*(x)\| \leq C', \; \forall\, n \in \mathbb{N}.$$

Let $\xi_j = \pm 1, \; j = 1, 2, ..., n$. Then, we obtain

$$\|\sum_{j=1}^{n} \xi_j b^*(a_j)a_j^*\| \leq \|\sum_{j=1}^{n} \varepsilon_j b^*(a_j)a_j^*\| + \|\sum_{j=1}^{n} \varepsilon_j^{'} b^*(a_j)a_j^*\|,$$

where $\varepsilon_j = 1$ and $\varepsilon_j^{'} = 0 \;$ if $\xi_j = 1, \;\; \varepsilon_j = 0$. Also, $\varepsilon_j^{'} = 1$ if $\xi_j = -1, \;\; j = 1, 2, ...n$, $n \in \mathbb{N}$.
Therefore, we get

$$\|\sum_{j=1}^{n} \xi_j b^*(a_j)b_j^*\| \leq \|\sum_{j=1}^{n} \xi_j b^*(a_j)a_j^*\| \leq C,$$

Thus, by Lemma ..... , the series $\sum_{j=1}^{\infty} b^*(a_j)b_j^*$ is $\sigma\,(\mathcal{B}_1^*, \mathcal{B}_1^{**})$- unconditionally Cauchy. Since $\mathcal{B}_1^*$ is weakly complete, it follows that $\sum_{j=1}^{\infty} b^*(a_j)b_j^*$ is $\sigma(\mathcal{B}_1^*, \mathcal{B}_1^{**})$ Convergent.
Also, for all $x \in \mathcal{B}_1$, we have

$$\sum_{j=1}^{\infty} b^*(a_j)b_j^*(x) = \sum_{j=1}^{\infty} b^*(a_j)a_j^* = a^*(x) = b^*(x).$$

Thus the series $\left\{\sum_{j=1}^{\infty} a^*(a_j)b_j^*\right\}$ is $\sigma(\mathcal{B}_1^*, \mathcal{B}_1^{**})$ convergent to b^*, since $b^* \in \mathcal{B}_1^*$ is arbitrary and $\mathcal{B}_1^*$ is separable, we arive at a contradiction.
Hence $\mathcal{B}$ has no Schauder frame satisfying property $\mathscr{U}$. □

Recall that a Banach space $\mathcal{B}$ is said to have property $\mathscr{P}$ if every subspace $\mathcal{B}_1^*$ of $\mathcal{B}^*$ with the property that for each b^* in $\mathcal{B}^*$ there exists a sequence $\{b_n^*\}$ in $\mathcal{B}_1^*$ such that for all $x \in \mathcal{B}$

$$b^*(x) = \lim_{n\to\infty} b_n^*(x)$$

and $\lim_{n\to\infty} b^{**}(b_n^*)$ exists is non-separable.

In the following result, we discuss the existence of a Schauder frame satisfying property $\mathscr{U}$ is separable Banach space containing a subspace having property $\mathscr{P}$.

Theorem 2.3. *If a separable Banach space $\mathcal{B}$ has a subspace satisfying property ($\mathscr{P}$), then $\mathcal{B}$ has no Schauder frame satisfying property $\mathscr{U}$.*

Proof. Let $(\{a_n\}, \{a_n^*\})$ $(\{a_n\} \subset \mathcal{B}, \{a_n^*\} \subset \mathcal{B}^*)$ is a Schauder frame satisfying property $\mathscr{U}$.
Let $\mathcal{B}_1 \subset \mathcal{B}$ be a subspace of $\mathcal{B}$ satisfying property $\mathscr{P}$.
For each n, let b_n^* be the restriction of a_n^* to $\mathcal{B}_1$.
Let $\mathcal{B}_0 = [b_n^*]$ and let $b^* \in \mathcal{B}_1^*$ be any arbitrary functional. Then, for each $a^* \in \mathcal{B}^*$, the series $\{\sum_{n=1}^{\infty} a^*(a_n)a_n^*\}$ is $\sigma(\mathcal{B}^*, \mathcal{B})$ - unconditionally convergent to a^* in $\mathcal{B}^*$.
Now, define a sequence $\{\alpha_n\}$ by $\alpha_k = \sum_{j=1}^{k} a^*(a_j)b_j^*$, $k \in \mathbb{N}$. Then $\{\alpha_n\} \in \mathcal{B}_0$ and for each b in $\mathcal{B}_1$, we have

$$\lim_{k\to\infty} \alpha_k(y) = \lim_{k\to\infty} \sum_{j=1}^{k} a^*(a_j)b_j^*(y) = b^*(y).$$

Then following the steps in the proof the Theorem... the series $\sum_{j=1}^{\infty} a^*(a_j)b_j^*$ is $\sigma(\mathcal{B}_1^*, \mathcal{B}_1^{**})$ - unconditionally Cauchy.
This implies $\lim_{k\to\infty} b^{**}(\alpha_k)$ exists for each $b^{**} \in \mathcal{B}_1^{**}$. This is a contradiction. □

Remark 2.4. As an application of the above results, one can verify that $L^1[0,1]$ has no Schauder frame satisfying property $\mathscr{U}$.

3. Conclusion

Schauder frames extend the concept of Schauder bases and play a fundamental role in the study of the structure and decomposition of elements in Banach spaces. Their flexibility makes them valuable tools in signal processing and data representation. In this paper, we defined and analysed property $\mathscr{U}$, a structural condition on Schauder frames, and investigated its existence across different Banach spaces. Further we identified key constraints that prevent the existence of Schauder frames satisfying property $\mathscr{U}$, particularly in separable Banach spaces containing subspaces with non-separable, weakly complete duals. These findings reveal intrinsic limitations that arise from the interplay between a space's topology and its dual structure. We further explored the class of Banach spaces possessing property ($\mathscr{P}$) and provided insight into how this property influences the feasibility of constructing Schauder frames that satisfy property $\mathscr{U}$. The analysis culminates in the demonstration that $L^1[0,1]$, a classical and extensively studied Banach space, does not admit any Schauder frame satisfying property $\mathscr{U}$—highlighting that even well-understood spaces can exhibit subtle and restrictive structural behaviour with respect to frames.

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