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A Short Note on Abstract Frames

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Abstract. Classical frame theory is traditionally formulated in Hilbert spaces and depends fundamentally on the existence of an inner product together with appropriate frame inequalities. Motivated by the reconstruction principle underlying these theories, this paper considers an elementary algebraic framework based solely on a pair of linear operators satisfying a factorization identity. The proposed notion of an Abstract Frame separates the reconstruction mechanism from metric assumptions and therefore applies to arbitrary vector spaces. Basic structural properties are established, including injectivity of the analysis operator, surjectivity of the synthesis operator, projection properties, and invariance under linear isomorphisms and product constructions. Several illustrative examples are presented together with a discussion showing how classical Hilbert frames naturally fit within this algebraic framework. The paper concludes by indicating possible extensions toward Banach, operator-valued, and categorical frame theories.

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1. Introduction

The representation of vectors through redundant families of generating elements is a cornerstone of modern analysis, offering essential robustness against perturbations in fields such as approximation theory and signal processing [12, 13]. Since the seminal work of Duffin and Schaeffer on nonharmonic Fourier series [8], classical frame theory has evolved into a sophisticated framework within Hilbert spaces, as detailed in several foundational texts and pedagogical introductions [7, 10, 17]. Recent advancements have further expanded this scope to include finite frames and their varied applications [4, 6], as well as specialized structures like woven K-frames in quaternionic Hilbert spaces [16].

A significant portion of frame research focuses on generalizing the analytic structure of the underlying space or the nature of the coefficient system. This includes the development of semi-frames and Hilbert scales [1], as well as the study of Bessel multipliers [2] and operator orbit frames [3]. Other generalizations have moved toward Banach spaces through Schauder frames [9] or have explored the decomposition of spaces into fusion frames [11], also known as frames of subspaces [5]. Furthermore, the theory has been extended to accommodate operator-valued frames [14], involving deep investigations into their localization [12] and their connection to positive operator-valued measures [15].

While these diverse generalizations often preserve the analytic and metric properties of the spaces involved, many of the essential reconstruction principles remain meaningful even in the absence of an inner product or norm . The present work is motivated by the observation that several familiar properties of classical frames are actually immediate consequences of the simple factorization identity $S \circ A = I_X$ rather than specific metric assumptions.

The purpose of this note is to describe a minimal algebraic setting that captures the essential features of a frame without relying explicitly on an inner product. By focusing on the reconstruction identity itself, we propose the notion of an Abstract Frame, which may serve as a common underlying framework for many of the existing generalized systems mentioned above. This algebraic approach allows for the investigation of fundamental properties—such as the injectivity of the analysis operator and the surjectivity of the synthesis operator—within arbitrary vector spaces.

2. Abstract Frames

Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let Λ be an index set. Suppose

$$A : X \to \mathbb{K}^\Lambda$$

is a linear analysis operator assigning to every $x \in X$ its coefficient sequence

$$A(x) = \{\alpha_\lambda(x)\}_{\lambda \in \Lambda},$$

where $\alpha_\lambda(x) \in \mathbb{K}$ for each $\lambda \in \Lambda$. Similarly let $S : \mathbb{K}^\Lambda \to X$ be a linear synthesis operator.

Definition 2.1. The pair (A, S) is called an *Abstract Frame* for X whenever

$$S \circ A = I_X.$$

Thus every vector admits the representation

$$x = S(Ax).$$

Note. One may notice that no topology or norm is required.

We shall now discuss some basic properties of analysis and synthesis operators associated with abstract frames.

Proposition 2.2. The analysis operator A is injective.

Proof. To prove that A is injective, we must show that whenever two elements of X have the same image under A, then the two elements are equal. Let $x, y \in X$ and suppose that

$$A(x) = A(y).$$

Since (A, S) is an Abstract Frame, we have

$$S \circ A = I_X.$$

Therefore, applying S to both sides of $A(x) = A(y)$ gives

$$S(A(x)) = S(A(y)).$$

Using the identity $S \circ A = I_X$, this becomes

$$I_X(x) = I_X(y).$$

Thus $x = y$. Hence, distinct vectors in X cannot have the same coefficient representation under A. Consequently, A is injective. □

Next, we show that an abstract frame uniquely determines coefficient representation inside the range of an analysis operator associated with an abstract frame.

Proposition 2.3. Every Abstract Frame gives a unique coefficient representation inside the range of A.

Proof. Let $c \in \mathrm{Ran}(A)$. By definition of the range of A, there exists at least one vector $x \in X$ such that $c = A(x)$. We now prove that such a vector is unique. Suppose that there are two vectors $x, y \in X$ satisfying $c = A(x) = A(y)$. Then $A(x) = A(y)$. By the injectivity of A, proved in Proposition 2.2, it follows that $x = y$. Therefore, each coefficient vector c belonging to $\mathrm{Ran}(A)$ comes from exactly one vector of X. Hence the coefficient representation is unique inside the range of the analysis operator. □

In the following proposition, we show that the synthesis operator is surjective.

Proposition 2.4. The synthesis operator S associated with an abstract frame is surjective.

Proof. To show that S is surjective, we must prove that for every element $x \in X$, there exists some coefficient vector $c \in \mathbb{K}^\Lambda$ such that

$$S(c) = x.$$

Let $x \in X$ be arbitrary. Since $A : X \to \mathbb{K}^\Lambda$, the vector

$$c = A(x)$$

belongs to $\mathbb{K}^\Lambda$. Using the defining identity of an Abstract Frame,

$$S \circ A = I_X,$$

we obtain

$$S(c) = S(A(x)) = I_X(x) = x.$$

Thus, every element $x \in X$ is the image under S of the coefficient vector $A(x)$. Therefore

$$\mathrm{Ran}(S) = X,$$

and hence S is surjective. □

Remark 2.5. Unlike classical frame theory, one may observe that the proofs of the results are obtained without the use of frame bounds or topological arguments. It follows solely from the factorization identity defining an Abstract Frame.

Next, we shall discuss the stability of abstract frames. Notice that, one of the attractive features of classical frames is stability. The same phenomenon occurs in the present setting. We prove the following result.

Theorem 2.6. *Let (A, S) be an Abstract Frame. If $T : X \to X$ is invertible, then*

$$(A \circ T^{-1}, T \circ S)$$

is also an Abstract Frame.

Proof. Since (A, S) is an Abstract Frame for X, we have

$$S \circ A = I_X.$$

Let

$$A_T = A \circ T^{-1}$$

and

$$S_T = T \circ S.$$

We need to prove that the pair (A_T, S_T) satisfies the defining identity of an Abstract Frame, namely

$$S_T \circ A_T = I_X.$$

Let $x \in X$ be arbitrary. Then

$$(S_T \circ A_T)(x) = S_T(A_T x).$$

Using the definitions of A_T and S_T, we get

$$S_T(A_T x) = (T \circ S)\big((A \circ T^{-1})(x)\big).$$

Hence

$$(S_T \circ A_T)(x) = T\left(S\left(A(T^{-1}x)\right)\right).$$

Since $S \circ A = I_X$, it follows that

$$S(A(T^{-1}x)) = T^{-1}x.$$

Therefore,

$$(S_T \circ A_T)(x) = T(T^{-1}x) = x.$$

Since this holds for every $x \in X$, we conclude that

$$S_T \circ A_T = I_X.$$

Thus

$$(A \circ T^{-1}, T \circ S)$$

is an Abstract Frame for X. □

Theorem 2.7. *Suppose (A, S) is an Abstract Frame. Then*

$$\ker(A) = \{0\},$$

and

$$X \cong \mathrm{Ran}(A).$$

Consequently, every Abstract Frame embeds the underlying vector space into its coefficient space.

Proof. Since (A, S) is an Abstract Frame, by definition we have

$$S \circ A = I_X.$$

First, we prove that

$$\ker(A) = \{0\}.$$

Let $x \in \ker(A)$. Then

$$A(x) = 0.$$

Applying S to both sides, we obtain

$$S(A(x)) = S(0).$$

Since S is linear,

$$S(0) = 0.$$

Using the identity $S \circ A = I_X$, we get

$$x = I_X(x) = S(A(x)) = 0.$$

Hence, every element of $\ker(A)$ is zero. Therefore,

$$\ker(A) = \{0\}.$$

Now we prove that

$$X \cong \mathrm{Ran}(A).$$

Consider the mapping

$$A : X \to \mathrm{Ran}(A), \qquad x \mapsto A(x).$$

This map is linear because A is linear. By definition of $\mathrm{Ran}(A)$, the map

$$A : X \to \mathrm{Ran}(A)$$

is surjective. Indeed, if $c \in \mathrm{Ran}(A)$, then there exists some $x \in X$ such that

$$c = A(x).$$

We have already shown that

$$\ker(A) = \{0\},$$

so A is injective. Therefore, A is a bijective linear mapping from X onto $\mathrm{Ran}(A)$.

Moreover, the inverse mapping is given by the restriction of S to $\mathrm{Ran}(A)$:

$$A^{-1} = S|_{\mathrm{Ran}(A)}.$$

Indeed, if $c \in \mathrm{Ran}(A)$, then there exists $x \in X$ such that

$$c = A(x).$$

Therefore

$$S(c) = S(A(x)) = x.$$

Thus $S|_{\mathrm{Ran}(A)}$ recovers the unique vector whose coefficient representation is c. Hence, A is a linear isomorphism between X and $\mathrm{Ran}(A)$, and consequently

$$X \cong \mathrm{Ran}(A).$$

Since $\mathrm{Ran}(A)$ is a subspace of the coefficient space $\mathbb{K}^\Lambda$, the analysis operator A identifies X with a subspace of its coefficient space. Thus every Abstract Frame embeds the underlying vector space into its coefficient space. $\square$

Definition 2.8. Two Abstract Frames (A_1, S_1) and (A_2, S_2) are called equivalent if there exists an invertible operator T such that

$$A_2 = A_1 T^{-1} \;\; \textit{and} \;\; S_2 = T S_1.$$

Note. This notion coincides with the usual equivalence of Hilbert-space frames generated by invertible operators.

3. Examples

Example 3.1. Let $X = \mathbb{R}^n$. Define

$$A(x) = x \quad and \quad S(x) = x.$$

Then, $SA = I$. So (A, S) is an Abstract Frame.

Example 3.2. Let $X = \mathbb{R}^2$. Define an operator A as

$$A(x_1, x_2) = (x_1, x_2, x_1 + x_2).$$

and the synthesis operator as

$$S(a, b, c) = (a, b).$$

Then $SA = I$. Thus, redundant coordinates naturally produce an Abstract Frame.

The flexibility of the definition allows many different constructions. Some of them are given as follows:

Example 3.3 (A Redundant Coordinate Representation)**.** Consider $X = \mathbb{R}^3$. Define

$$A(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1 + x_2, x_2 + x_3).$$

Suppose that

$$S(a, b, c, d, e) = (a, b, c).$$

Then, one can verify that

$$SA = I_X,$$

showing that additional redundant coordinates do not destroy the reconstruction property.

Example 3.4 (Polynomial Spaces)**.** Let

$$P_2 = \left\{p(x) = a_0 + a_1 x + a_2 x^2 : \; a_0, a_1, a_2 \in \mathbb{K}\right\},$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Thus, P_2 is the three-dimensional vector space of all polynomials of degree at most two.

Equip P_2 with the inner product

$$\left\langle a_0 + a_1 x + a_2 x^2, \; b_0 + b_1 x + b_2 x^2 \right\rangle = a_0 \overline{b_0} + a_1 \overline{b_1} + a_2 \overline{b_2},$$

where the bar denotes complex conjugation (and may be omitted when $\mathbb{K} = \mathbb{R}$). With this inner product, P_2 is a three-dimensional Hilbert space.

Let

$$X = P_2.$$

Define the analysis operator

$$A : P_2 \longrightarrow \mathbb{K}^4$$

by

$$A(p) = \big(p(0), \, p(1), \, p(2), \, p'(0)\big).$$

Since these four functionals determine every polynomial in P_2, there exists a synthesis operator recovering p uniquely.

Consequently,

$$SA = I_{P_2},$$

and (A, S) forms an Abstract Frame.

Example 3.5 (Finite-Dimensional Hilbert Spaces)**.** Every finite frame

$$\{f_1, \ldots, f_m\}$$

for $\mathbb{R}^n$ naturally determines an Abstract Frame by defining the analysis operator

$$A(x) = (\langle x, f_1\rangle, \ldots, \langle x, f_m\rangle),$$

while the synthesis operator is given through any dual frame. Thus, the present framework extends all finite frame systems without reference to frame bounds.

The preceding examples illustrate that abstract frames arise naturally in finite-dimensional linear algebra, interpolation theory, and classical frame theory. The definition is sufficiently general to encompass both redundant and non-redundant coordinate systems while retaining a universal reconstruction formula.

Remark 3.6. The Abstract frames are related to classical frames as follows:
Let H be a Hilbert space and $\{f_n\}_{n=1}^{\infty}$ be a frame. Consider the analysis operator defined as $A(x) = \{\langle x, f_n\rangle\}$, and the synthesis operator defined as

$$S(c_n) = \sum_{n=1}^{\infty} c_n \widetilde{f}_n,$$

where $\{\widetilde{f}_n\}$ denotes the canonical dual frame. Then the reconstruction formula $S(Ax) = x$ shows that every Hilbert-space frame is an Abstract Frame.

4. Conclusion

The notion of an Abstract Frame introduced here isolates the fundamental reconstruction property from the additional metric structure usually imposed in frame theory. Several basic algebraic properties follow naturally from the identity $SA = I$. The framework encompasses ordinary Hilbert frames while remaining independent of norms and inner products. Future work may incorporate topological Banach space, and operator-valued structures into this abstract setting.

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