Poincare Journal of Mathematical Physics and Applications
Vol. 1, No. 2 (2026), 117-133
©Poincare Publishers
DOI: 10.46753/pjmpa.2026.v01i02.003
Online Published on 27.11.2026

Notes on sharp theorems on multipliers in harmonic and analytic function spaces in higher dimension and related problems

R. F. Shamoyan[1,†], and M. G. Bashmakova[2]

[1]Department of Mathematics, Bryansk State Technical University, Bryansk, Russia.
rsham@mail.ru
[2]Department of Mathematics, Bryansk State Technical University, Bryansk, Russia.
bashmakova@example.com

Submitted: 15.05.2026 | Revised: 06.09.2026 | Accepted: 09.09.2026

Abstract. Coefficient multipliers play an important role in the study of harmonic and analytic function spaces. The sharp multiplier results of Shields and Williams for harmonic functions in the unit disk provided a foundation for subsequent developments in higher-dimensional settings. This review presents a systematic account of the evolution of coefficient multiplier theory for harmonic functions of several variables, with particular emphasis on spaces defined on the unit ball and the use of spherical harmonics in obtaining sharp characterizations. We survey major results concerning multiplier sequences for different classes of harmonic functions and discuss their extensions to various function spaces and domains.

The article also considers coefficient multipliers in analytic function spaces of several variables, placing these developments in the context of corresponding one-dimensional theories and classical results. Particular attention is given to sharp multiplier conditions, the role of growth and smoothness properties, and the connections between harmonic and analytic settings. Results obtained by several authors are organized to highlight the common principles underlying these theories as well as their significant differences. Several open problems and possible directions for future research are also formulated, particularly for harmonic and analytic function spaces in higher dimensions and more general domains. The aim is to provide a unified and accessible perspective on the subject and to stimulate further developments in this area of harmonic analysis and function-space theory.

Keywords: Coefficient multipliers, harmonic functions, spherical harmonics, unit ball, unit polydisk
MSC (2020): 30H25, 31B05, 42B35
Communicated by: Shiv K. Kaushik
[†]**Corresponding author**

1. Introduction

The well - known problem of description of spaces of coefficient multipliers of various pairs of analytic functions was considered as far as we know in the unit disk for the first time in mathematical literature in classical papers of G. H. Hardy and J.E. Littlewood. The formulation of problem is very simple. Let X, Y be two spaces of analytic functions in the unit disk. We fix an arbitrary analytic f function in X with $\{c_k\}$ Taylor coefficients and we wish to describe all $\{a_k\}$ sequences such that a new analytic g function with $\{c_k \times a_k\}$ Taylor coefficients belong to that mentioned fixed Y analytic function space in the unit disk. Note that, this simple problem of description of spaces of coefficient multipliers for various pairs of spaces (X, Y) can be similarly and very easily passed to the case of several complex variables namely to the so called unit polydisk case in $\mathbb{C}^n$. We leave this very easy task to readers. The main and only ingredient here is to change for example simply $\{c_k\}$ sequence to $c_{k_1,\dots,k_m}$ sequence in our definition above in the unit disk, using appropriate Taylor expansion of analytic function in the unit polydisk. The same problem of descriptions of coefficient multipliers may be considered in various interesting analytic function spaces which can be also defined there in the unit polydisk. Moreover we can easily pass this rather simple and transparent definition which we provided for analytic function spaces in the unit disk and polydisk in $\mathbb{C}$ and $\mathbb{C}^n$ to so called well studied by many authors previously harmonic function spaces in the unit disk in a complex plane $\mathbb{C}$. And then also to try to solve this problem of finding of sharp estimates of spaces of coefficient multipliers for various pairs of harmonic function spaces in the unit disk. Note, that many interesting sharp one dimensional results were obtained during several decades by many authors for harmonic and analytic spaces in the unit disk. We, for example, wish to mention interesting papers of P. Duren [18] , M. Pavlovic and M. Mateljevic [24],[29], O. Blasco [10],[11], M. Jevtic [22], D. Williams and A. Shields [44], M. Arsenović and M. Vukotic [23] and their coauthors and also many others.

The situation with the problem of descriptions of spaces of coefficient multipliers for various pairs of spaces in $\mathbb{C}^n$ is technically much more complicated, in these spaces of analytic and harmonic functions of several variables. In $\mathbb{C}^n$ many such problems exists, and in this direction of research many problems are completely open and this topic may be considered as a new research field in function space theory of several variables. We mention here separately a very interesting paper of Marek Nawrocki [25] where the problem of description of spaces of coefficient multipliers in area Analytic Nevanlinna type spaces were solved in the polydisk with rather difficult technique. We collect these new sharp results in the second part of this expository. Note with various interesting new short remarks and various short comments concerning this issue. In the first part, we collect recent sharp results obtained in various harmonic function spaces of several variables in $\mathbb{C}^n$, established by R.Shamoyan [35] and in collaboration with M. Arsenović [4]-[7]. This problem is completely new. First sharp results were obtained many decades ago by D. Williams and A. Shields [44]. In harmonic spaces of several variables this problem stood open for many decades. In the paper [35], R. F. Shamoyan established the first such result. Then later he managed to develop these his ideas later to obtain a long series of similar sharp results on coefficient multipliers for other function spaces in $\mathbb{C}^n$ of harmonic function spaces of several variables in $\mathbb{C}^n$.

We collect all these interesting results in this note, provided earlier in different papers of the author for spaces of several variables. In the first part we collect new sharp theorems in harmonic spaces of several variables then new sharp results in analytic spaces of several

variables. We mention here also that first we collect definitions of objects then theorems. Some similar very interesting new results in harmonic spaces of several variables were obtained recently by E. Malinnikova and K. Eikrem [20].

At the end of this section we would like to mention several important old research papers closely related to the topic of this note. Harmonic functions and spaces of harmonic functions of several variables in various domains were studied in important papers of L. Carleson, E. Stein, G. Weiss and others in [47], [**?**], [15], [45]. We also mention important papers of O. Oleinik, and B. Pavlov (see [26] and [27]) concerning topic of this paper, where various embedding theorems in harmonic function spaces were provided and various references there. We also would like to refer the readers to [13], [21] for issues regarding boundary behaviour of harmonic functions. We note, that spherical harmonics which we use heavily in this paper and which serve as a base for this paper were also used for example in [17].

As far as we know the theory of Harmonic function spaces of several variables in various domains in $\mathbb{C}^n$ started to develop in papers of E. Stein and G. Weiss and later by various other authors and there are many interesting open problems in this research field. In addition various new interesting discussions and questions in this direction of research will be also provided and posed by authors. Finally related results on coefficient multipliers in harmonic function spaces were given in papers of R. Timoney and D. Stegenga.

In various spaces of analytic functions of one variable coefficient multipliers theorems (sharp results) can be seen in various old interesting papers of M. Jevtic, M. Pavlovic (see [22],[29]), M. Mateljevic [24], D. Vukotic [23], we refer the reader to those research papers, posing as a problem to expand such one-dimensional results to the case of the unit polydisk in $\mathbb{C}^n$.

The study of spaces of Coefficient multipliers of harmonic function spaces in several variables is a relatively new research field and may have various interesting applications in function theory of several variables and in harmonic function space theory of several variables. Various very interesting problems in various spaces of harmonic functions of several variables were considered by various authors during last several decades in [1]-[14].

We wish to mention that our Spherical harmonics were also previously used in [19] in a very elegant manner to construct very special Bergman type kernels for the study of Bergman spaces of harmonic functions of several variables in the unit ball in $\mathbb{R}^n$. Several questions with the help of these interesting kernels of Bergman type were completely solved at the same time extending well known one-dimensional results. Descriptions of continuous functional were given, bounded projections were provided in A_p spaces of Bergman of harmonic functions of several variables in the unit ball and the action of the operator of fractional integration was also provided in these spaces. We refer the reader to [19] for these results and also various references there. However issues related to the coefficient multipliers which we considered in this note for A_p Bergman spaces of harmonic functions in the unit ball B_n were not considered there.

We omit almost everywhere below various detailed comments concerning approaches which we used in proofs of our results referring the reader to mentioned papers. All our proofs are based on standard and known estimates of complex and harmonic function theory of several variables in the unit ball and in the unit polydisk.

We formulate some perspectives of topics we considered in this note and interesting new problems. First we didn't consider spaces of harmonic functions of several variables of BMOA

type in the unit ball . As far as we know these harmonic classes are completely new and the problem of description of spaces of coefficient multipliers may be considered here. Another new interesting problem to describe spaces of coefficient multipliers of harmonic function spaces in upper half space, such function spaces were considered by many authors previously but the topic of multipliers as far as we know is completely new for such classes.
Note that in [20] (see also the references therein), spherical harmonics, which are important for this paper, were used intensively in various ways to study properties of Bergman-type spaces of harmonic functions in the unit ball.

2. Main results on coefficient multipliers

The goal of this section, the main section of this paper, is to collect various new sharp results from papers of first a author concerning coefficient multipliers of spaces of harmonic several variables and also at the end of this section we provide certain new sharp results on coefficient multipliers in analytic spaces in the unit polydisk. The role of spherical harmonics are very important in this paper and we shortly for convenience of readers provide also below some details concerning these interesting objects. We define various spaces of harmonic functions of several variables below and then after that provide the list of our sharp results on coefficient multipliers in harmonic function spaces in several variables. At the end of this note some new results (sharp theorems) on coefficient multipliers in analytic function spaces of several variables will be also provided by us. These results were also recently obtained by R. Shamoyan.

We first provide definitions of basic objects of harmonic function spaces in the unit ball and then we define below various standard spaces of harmonic functions in the unit ball, they were studied previously by various authors in recent decades.

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^n, \mathbb{S}^n = \partial\mathbb{B}$ is the unit sphere in $\mathbb{R}^n$. We define as usual by dx' the normalized Lebeques measure on the unit sphere $\mathbb{S}$. For $x \in \mathbb{R}^n$ we have $x = rx'$, where $r = |x| = \sqrt{\sum_{j=1}^{n} x_j^2}$ and $x' \in \mathbb{S}$. We denote by $dx = dx_1...dx_n = r^{n-1}drdx'$ normalized Lebesgue measure on $\mathbb{B}$ so that $\int_{\mathbb{B}} dx = 1$ and by $h(\Omega)$ - the space of all harmonic functions in an open set Ω.

Spaces of harmonic functions worth mixed-norm, Bergman type, Bloch type and Weighted Hardy type in the unit ball were studied by many authors in recent decades, we mention for example various research papers of A. Aleksandrov, P. Kargaev, A. E. Djrbashian and many others. The problem of description of spaces of coefficient multipliers of these harmonic function classes of several variables was an open problem for many decades.

Harmonic weighted Bergman spaces $A^p_\alpha(\mathbb{B})$ on $\mathbb{B}$ are defined for $\alpha > -1$ and $0 < p \leq \infty$ by

$$A^p_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{p,\alpha} = \Big(\int_{\mathbb{B}} |f(rx')|^p (1-r)^\alpha r^{n-1} dr dx' \Big)^{\frac{1}{p}} < \infty \right\},$$

$$A^\infty_\alpha(\mathbb{B}) = \left\{ f \in h(\mathbb{B}) : \|f\|_{\infty,\alpha} = \sup_{x \in \mathbb{B}} |f(x)|(1-|x|)^\alpha < \infty \right\}.$$

It is easy to see that the spaces $A^p_\alpha = A^p_\alpha(\mathbb{B})$ are Banach spaces for $1 \leq p \leq \infty$ and complete metric spaces for $0 < p < 1$.

Spherical harmonics which may be considered in some sense as a base of this note were studied probably for the first time in well known papers of E. Stein in a well known monograph of E. Stein and G. Weiss and the Bergman A_p spaces of harmonic functions of several variables in the unit ball in $\mathbb{R}^n$ were considered for the first time as far as we know by R. Coifman and R. Rochberg. These issues can be seen in [16] and [46].

For $0 < p < \infty, 0 \leq r < 1$ and $f \in h(\mathbb{B})$ we set

$$M_p(f,r) = \left(\int\limits_{\mathbb{S}} |f(rx')|^p dx' \right)^{\frac{1}{p}},$$

with the usual modification in case $p = \infty$. Weighted Hardy spaces are defined, for $\alpha \geq 0$ and $0 < p \leq \infty$, by

$$H^p_\alpha(\mathbb{B}) = H^p_\alpha = \left\{ f \in h(\mathbb{B}) : \|f\|_{p,\alpha} = \sup_{r<1} M_p(f,r)(1-r)^\alpha < \infty \right\}.$$

For $\alpha = 0$ the space H^p_α is denoted simply by H^p.

For $0 < p \leq \infty, 0 < q \leq \infty$ and $\alpha > 0$ we set mixed (quasi)-norms $\|f\|_{p,q;\alpha}$ as

$$\|f\|_{p,q;\alpha} = \left(\int\limits_0^1 M_q(f,r)^p (1-r^2)^{\alpha p - 1} r^{n-1} dr \right)^{\frac{1}{p}}, f \in h(\mathbb{B}), \tag{2.1}$$

with the conventional interpretation for $p = \infty$, and the corresponding spaces

$$B^{p,q}_\alpha(\mathbb{B}) = B^{p,q}_\alpha = \{ f \in h(\mathbb{B}) : \|f\|_{p,q;\alpha} < \infty \}.$$

These spaces are complete metric spaces and for $\min(p,q) \geq 1$ they are Banach spaces.

Such type harmonic spaces in the ball and in the unit disk were studied in [1]-[3],[46],[44],[19].

Definition 1. Let $0 < p, q < \infty$ and $\alpha > 0$. The harmonic Triebel-Lizorkin space $F^{p,q}_\alpha(\mathbb{B}) = F^{p,q}_\alpha$ consists of all functions $f \in h(\mathbb{B})$ such that

$$\|f\|_{F^{p,q}_\alpha} = \left(\int\limits_{\mathbb{S}} \left(\int\limits_0^1 |f(rx')|^p (1-r)^{\alpha p - 1} dr \right)^{\frac{q}{p}} dx' \right)^{\frac{1}{q}} < \infty$$

Such type harmonic spaces in the ball and in the unit disk were studied in [1]-[3],[46],[44],[19].
Next we need certain facts on spherical harmonics and Poisson kernel. Let $Y^{(k)}_j$ be the spherical harmonics of order k, $1 \leq j \leq d_k$, on $\mathbb{S}^n$. Next,

$$Z^{(k)}_{x'}(y') = \sum_{j=1}^{d_k} Y^k_j(x') \overline{Y^k_j(y')}$$

are zonal harmonics of order k. Note that, the spherical harmonics $Y^{(k)}_j (k \geq 0, 1 \leq j \leq d_k)$ form an orthonormal basis of $L^2(\mathbb{S}, dx')$. Every $f \in h(\mathbb{B})$ has an expansion

$$f(x) = f(rx') = \sum_{k=0}^{\infty} r^k b_k \cdot Y^k(x'),$$

where $b_k = (b_k^1, ..., b_k^{d_k}), Y^k = (Y_1^{(k)}, ..., Y_{d_k}^{(k)})$ and $b_k \cdot Y^k$ is interpreted in the scalar product sense: $b_k \cdot Y^k = \sum\limits_{j=1}^{d_k} b_k^j Y_j^{(k)}$. We often write, to stress dependence on a function $f \in h(\mathbb{B}), b_k = b_k(f)$ and $b_k^j = b_k^j(f)$, in fact we have linear functionals $b_k^j, k \geq 0, 1 \leq j \leq d_k$ on the space $h(\mathbb{B})$.

We denote the Poisson kernel for the unit ball by $P(x, y')$, it is given by

$$P(x, y') = P_{y'}(x) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{d_k} Y_j^{(k)}(y') Y_j^{(k)}(x') =$$

$$= \frac{1}{nw_n} \frac{1 - |x|^2}{|x - y'|^n}, x = rx' \in \mathbb{B}, y' \in \mathbb{S}^n,$$

where w_n is the volume of the unit ball in $\mathbb{R}^n$. We are going to use also a Bergman kernel for A_β^p spaces, this is the following function

$$Q_\beta(x, y) = 2 \sum_{k=0}^{\infty} \frac{\Gamma(\beta + 1 + k + \frac{n}{2})}{\Gamma(\beta + 1)\Gamma(k + \frac{n}{2})} r^k \rho^k Z_{x'}^{(k)}(y'), x = rx', y = \rho y' \in \mathbb{B}, \beta \geq 0. \tag{2.2}$$

We recall the following definitions from [4].These definitions of Bergman and Poisson kernel are well known for experts.

Definition 2.

$$c = \{c_k^j : k \geq 0, 1 \leq j \leq d_k\}$$

and a harmonic function $f(rx') = \sum\limits_{k=0}^{\infty} \sum\limits_{j=1}^{d_k} r^k b_k^j(f) Y_j^{(k)}(x')$ we define

$$(c * f)(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k c_k^j b_k^j(f) Y_j^{(k)}(x'), rx' \in \mathbb{B},$$

if the series converges in $\mathbb{B}$. Similarly we define convolution of $f, g \in h(\mathbb{B})$ by

$$(f * g)(rx') = \sum_{k=0}^{\infty} \sum_{j=1}^{d_k} r^k b_k^j(f) b_k^j(f) Y_j^{(k)}(x'), rx' \in \mathbb{B}.$$

It is easily seen that $f * g$ is defined and harmonic in $\mathbb{B}$.

We define an operator of fractional integration for harmonic function in the unit ball. This definition is also crucial for this paper, we should note, that this operator was used by many authors previously in the theory of spaces of harmonic functions of several variables.

Definition 3. For $t > 0$ and a harmonic function $f(x) = \sum\limits_{k=0}^{\infty} b_k(f) Y^k(x')$ on the unit ball we define a fractional derivative of order t of f by the following formula:

$$(\Lambda_t f)(x) = \sum_{k=0}^{\infty} r^k \frac{\Gamma(k + n/2 + t)}{\Gamma(k + n/2)\Gamma(t)} b_k(f) \cdot Y^k(x'), x = rx' \in \mathbb{B}.$$

We define finally below so-called spaces of coefficient multipliers in harmonic function spaces of several variables based on definition 1.

Definition 4. Let X and Y be subspaces of $h(\mathbb{B})$. We say that a double indexed sequence c

is a multiplier from X to Y if $c * f \in Y$ for every $f \in X$. The vector space of all multipliers from X to Y is denoted by $M_H(X, Y)$.

This topic and this definition of coefficient multipliers in harmonic spaces in the unit ball via spherical harmonics are novel, they invented for the first time in papers of first author in 1998 in [36].

We collect a series of recent sharp results on coefficient multipliers in harmonic function spaces in the unit ball, obtained in the papers [4]-[32]. For precise definitions of the function spaces appearing in the statements of the theorems, which are not included in this paper, we refer the reader to [4]-[32].

Theorem 2.1. [4] *Let $1 < p \le q \le \infty$ and $m > \alpha - 1$. Then for a double indexed sequence $c = \{c_k^j : k \ge 0, 1 \le k \le d_k\}$ the following conditions are equivalent:*
1. $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,1})$.

2.The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition

$$N_1(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{\beta-\alpha+m+1} \int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from $B_\alpha^{p,1}$ to H_β^s.

Theorem 2.2. [4] *Let $\beta \ge 0, 0 < p \le 1, s \ge 1$ and $m > \alpha - 1$. Then, for a double indexed sequence $c = \{c_k^j : k \ge 0, 1 \le k \le d_k\}$ following two conditions are equivalent:*
1. $c \in M_H(B_\alpha^{p,1}, H_\beta^s)$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$N_s(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{m+1-\alpha+\beta} \left(\int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers $B_\alpha^{p,t}$ from to H_β^s.

Theorem 2.3. [8] *Let $0 < t, p \le 1, 1 \le s \le \infty$ and $m > \max(\alpha + \frac{n-1}{t} - n, -1)$.*
Then for a double indexed sequence $c = \{c_k^j : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:
1. $c \in M_H(B_\alpha^{p,t}, H_\beta^s)$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$T_s(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{\beta-\alpha+m+n-\frac{n-1}{t}} \left(\int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

For $1 < t \le \infty$ condition 1 implies condition 2.

Further we consider multipliers in spaces $F^{p,t}_{\alpha}, H^s_{\beta}$.

Theorem 2.4. [8] *Let $0 < t \le p \le 1 \le s \le \infty$ and $m > \max(\alpha + \frac{n-1}{t} - n, -1)$.*
Then for a double indexed sequence $c = \{c^j_k : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:
1. $c \in M_H(F^{p,t}_{\alpha}, H^s_{\beta})$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c^j_k Y^{(k)}_j(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$T_s(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{\beta-\alpha+m+n-\frac{n-1}{t}} \left(\int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from H^t_{α} to H^s_{β}.

Theorem 2.5. [8] *Let $0 < t \le 1 \le s \le \infty, \alpha \ge 0, m > \max(\alpha + \frac{n-1}{t} - n, -1)$ and $\beta > 0$.*
Then for a double indexed sequence $c = \{c^j_k : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:
1. $c \in M_H(H^t_{\alpha}, H^s_{\beta})$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c^j_k Y^{(k)}_j(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$T_s(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{\beta-\alpha+m+n-\frac{n-1}{t}} \left(\int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')|^s dx' \right)^{\frac{1}{s}} < \infty.$$

The following result concerns multipliers into Triebel-Lizorkin spaces.

Theorem 2.6. [8] *Let $0 < p \le 1 \le q \le \infty$ and $m > \alpha - 1$.*
Then for a double indexed sequence $c = \{c^j_k : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:
1. $c \in M_H(B^{p,1}_{\alpha}, F^{q,1}_{\beta})$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c^j_k Y^{(k)}_j(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$N_1(g) = \sup_{0\le\rho<1} \sup_{y'\in\mathbb{S}} (1-\rho)^{\beta-\alpha+m+1} \int\limits_{\mathbb{S}} |\Lambda_{m+1}(g * P_{x'})(\rho y')| dx' < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from $B^{p,1}_{\alpha}$ to $B^{q,s}_{\beta}$.

Theorem 2.7. [8] *Let $1 \le p \le q \le \infty, 1 \le s \le \infty$ and $m > \alpha - 1$.*
Then for a double indexed sequence $c = \{c^j_k : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:
1. $c \in M_H(B^{p,1}_{\alpha}, B^{q,s}_{\beta})$.

2. The function $g(x) = \sum\limits_{k\ge 0} r^k \sum\limits_{j=1}^{d_k} c^j_k Y^{(k)}_j(x')$ is harmonic in $\mathbb{B}$ and satisfies the following

condition:

$$N_s(g) = \sup_{0\le\rho<1}\sup_{y'\in\mathbb{S}}(1-\rho)^{\beta-\alpha+m+1}\left(\int_{\mathbb{S}}|\Lambda_{m+1}(g*P_{x'})(\rho y')|^s dx'\right)^{\frac{1}{s}} < \infty.$$

We define N_∞ as usual replacing L^p norm in N_p by L_∞ norm.

In the the following theorem we provide descriptions of spaces of multipliers from $B_\alpha^{p,1}$ to $B_\beta^{q,\infty}$.

Theorem 2.8. [8] *Let $0 < p \le q \le \infty$ $m > \alpha - 1$. Then for a double indexed sequence $c = \{c_k^j : k \ge 0, 1 \le j \le d_k\}$ the following conditions are equivalent:*
1. $c \in M_H(B_\alpha^{p,1}, B_\beta^{q,\infty})$.

2. The function $g(x) = \sum\limits_{k\ge0} r^k \sum\limits_{j=1}^{d_k} c_k^j Y_j^{(k)}(x')$ is harmonic in $\mathbb{B}$ and satisfies the following condition:

$$N_\infty(g) < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from H^s to A_β^∞.

Theorem 2.9. [7] *Let $1 < s < \infty, \beta > 0$ and let s' be the exponent conjugate to s. Then $c \in M_H(H^s, A_\beta^\infty)$ if and only if the function $g = g_c$ satisfies the following condition*

$$M_{s'}(g) = \sup_{0\le\rho<1}\sup_{y'\in\mathbb{S}}(1-\rho)^{\beta}\left(\int_{\mathbb{S}}|(g*P_{x'})(\rho y')|^{s'}dx'\right)^{\frac{1}{s}} < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from $B_\alpha^{p,q}$ to A_β^∞.

Theorem 2.10. [7] *Let $1 < p < \infty, 1 \le q < \infty, \alpha, \beta > 0$ and let q' be the exponent conjugate to q. Assume $m > \max(\alpha - \beta - 1, \beta - 1)$. Then $c \in M_H(B_\alpha^{p,q}, A_\beta^\infty)$ if and only if the function $g = g_c$ satisfies the following condition*

$$N_{q'}(g) = \sup_{0\le\rho<1}\sup_{y'\in\mathbb{S}}(1-\rho)^{m+1+\beta-\alpha}\left(\int_{\mathbb{S}}|\Lambda_{m+1}(g*P_{x'})(\rho y')|^{q'}dx'\right)^{\frac{1}{q'}} < \infty.$$

In the the following theorem we provide descriptions of spaces of multipliers from $A_\alpha^{p,1}$ to $A_\beta^{p,1}$.

Theorem 2.11. [32] *Let $g(x)$ be a harmonic function on $\mathbb{B}$ and*

$$g(x) = \sum_{k\ge0} r^k \sum_{j=1}^{d_k} c_k^{(j)} Y_j^{(k)}(x'), x = \rho x', \rho \in (0,1), x' \in \mathbb{S}^n.$$

Let $0 < p \le 1, \alpha \in (0,1), m > \max\{\alpha - 1, \frac{1}{p} - 1\}, \beta > 0$. Then the following assertions are equivalent:
1. $c = \{c_k^j : j = 1, ..., d_k, k \in \mathbb{Z}^+\} \in M_H(A_\alpha^{p,1}, A_\beta^{p,1})$.

2. $\sup\limits_{0<\rho<1}\sup\limits_{y'\in S^n}\left(\int\limits_{S^n}|\Lambda_{m+1}(g*P_{x'})(\rho y')|dx'\right)(1-\rho)^{m+1-\alpha+\beta} < \infty.$

In new sharp several assertions below we provide similar sharp theorems concerning coefficient multipliers but in analytic function spaces of several variables namely in the unit polydisk in $\mathbb{C}^n$ and provide certain new sharp results in this new interesting research field. In one-dimensional case these such type sharp results can be seen as well-known classical topic in complex function theory in the unit disk. First such sharp results on multipliers in the unit disk were provided by Hardy, Littlewood, Flett, Duren, Pavlovic and their various coauthors previously in recent decades and many years ago. All theorems of this note on coefficient multipliers in analytic spaces in the unit polydisk which we indicate below and formulate below are taken from research papers of R. Shamoyan.

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}, T = \partial U = \{z \in \mathbb{C} : |z| = 1\}, U^n$ is the unit polydisc in $\mathbb{C}^n, T^n \subset \partial U^n$ is the distinguished boundary of $U^n, \mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}, \mathbb{Z}_+^n$ is the set of all multi indexes and $I = [0, 1)$.

We use the following notation: for $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $k = (k_1, ..., k_n) \in \mathbb{Z}_n^+$ we set $z^k = z_1^{k_1}...z_n^{k_n}$ and for $(z_1, .., z_n) \in U^n$ and $\gamma \in \mathbb{R}$ we set $(1-|z|)^\gamma = (1-|z_1|)^\gamma...(1-|z|)^\gamma$ and $(1-z)^\gamma = (1-z_1)^\gamma...(1-z_n)^\gamma$. Next for $z \in \mathbb{R}^n, w \in \mathbb{C}^n$ we set $wz = (w_1z_1, ..., w_nz_n)$. Also for $k \in \mathbb{Z}_+^n$ and $a \in \mathbb{R}$ we set $k + a = (k_1 + a, ..., k_n + a)$. For $z \in \mathbb{C}^n$ we set $\bar{z} = (\bar{z}_1, ..., \bar{z}_n)$. For $k \in (0, \infty)^n$ we set $\Gamma(k) = \Gamma(k_1)...\Gamma(k_n)$.

The Lebesgue measure on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is denoted by $dV(z)$, normalized Lebesgue measure on T^n is denoted by $d\xi = d\xi_1...d\xi_n$ and $dR = dR_1...dR_n$ is the Lebesgue measure on $[0, \infty)^n$.

The space of all functions holomorphic in U^n we denote by $H(U^n)$. Every $f \in H(U^n)$ admits an expansion $f(z) = \sum\limits_{k \in \mathbb{Z}_+^n} a_k z^k$. For $\beta > -1$ the operator of fractional differentiation is defined by

$$D^\beta f(z) = \sum_{k \in \mathbb{Z}_+^n} \frac{\Gamma(k + \beta + 1)}{\Gamma(\beta + 1)\Gamma(k + 1)} a_k z^k, z \in U^n.$$

We can define the operator of fractional derivative for $\beta-$vector, $\beta = (\beta_1, ..., \beta_n), \beta_j > -1, j = 1, ..., n$.

$$D^{\vec{\beta}} f(z) = \sum_{k \in \mathbb{Z}_+^n} \frac{\Gamma(k_j + \beta_j + 1)}{\Gamma(\beta_j + 1)\Gamma(k_j + 1)} a_{k_1...k_n} z_1^{k_1}...z_n^{k_n}.$$

For $f \in H(U), 0 < p < \infty$ and $r \in I^n$ we have

$$M_p(f, r) = \left(\int\limits_{\mathbb{T}^n} |f(r\xi)|^p d\xi \right)^{\frac{1}{p}}, d\xi = d\xi_1...d\xi_n$$

with the conventional modification in the case $p = \infty$. For $0 < p \leq \infty$ we set standard analytic Hardy classes in the polydisc:

$$H^p(U^n) = \left\{ f \in H(U^n) : \|f\|_{H_h} = \sup_{r \in I^n} M_p(f, r) < \infty \right\}.$$

For $0 < p \leq \infty, 0 < q < \infty$ and $\alpha > 0$ we have mixed (quasi) norm spaces, defined as follows

$$A_\alpha^{p,q}(U^n) = \left\{ f \in H(U^n) : \|f\|_{A_\alpha^{p,q}}^q = \int\limits_{I^n} M_p^q(f, R)(1 - R)^{\alpha q - 1} dR < \infty \right\}.$$

If we replace the integration by I^n above by integration by unit interval I then we obtain other similar to these $A^{p,q}_{\alpha}$ analytic spaces but on subframe and we denote them by $B^{p,q}_{\alpha}$. For $0 < p, q < \infty$ and $\alpha > 0$ Lizorkin-Triebel spaces $F^{p,q}_{\alpha}(U^n) = F^{p,q}_{\alpha}$ consisting of all $f \in H(U^n)$ such that

$$\|f\|^p_{F^{p,q}_{\alpha}} = \int\limits_{T^n} \left(\int\limits_{I^n} |f(R\xi)|^q (1-R)^{\alpha q - 1} dR \right)^{\frac{p}{q}} d\xi < \infty.$$

If we replace the integration by I^n above by integration by unit interval I then we get other similar to these $F^{p,q}_{\alpha}$ analytic spaces, but on subframe and we denote them by $T^{p,q}_{\alpha}$.

For all $\alpha \geq 0, \beta \geq 0$ we set

$$A^{\infty,\infty}_{\alpha,\beta}(U^n) = \left\{ f \in H(U^n) : \|f\|_{A^{\infty,\infty}_{\alpha,\beta}} = \sup_{r \in I^n} (M_{\infty}(D^{\alpha} f, r))(1-r)^{\beta} < \infty \right\}.$$

We note, that replacing q by ∞ in a usual way give us some other spaces $F^{p,\infty}$. The limit space case $F^{p,\infty,s}(U^n)$ is defined as a space of all analytic functions f in the polydisc such that the function $\phi(\xi) = \sup_{r \in I^n} |f(r\xi)|(1-r)^s, \xi \in T^n$ is in $L^p(T^n, d\xi), s \geq 0, 0 < p \leq \infty$.

Ultimately, the limit space case $A^{p,\infty,s}(U^n)$ is the space of all analytic functions f in the polydisc such that

$$\sup_{r \in I^n} M_p(f, r)(1-r)^s < \infty, s \geq 0, 0 < p \leq \infty.$$

It is clear that the limit $F^{p,\infty,s}$ space is embedded in the last space we defined.

The following definition of coefficient multipliers is well known in the unit disk.

Definition 5. Let X and Y be quasi-normed subspaces of $H(U^n)$. A sequence $c = \{c_k\}_{k \in \mathbb{Z}^n_+}$ is said to be a coefficient multiplier from X to Y if for any function $f(z) = \sum\limits_{k \in \mathbb{Z}^n_+} a_k z^k$ in X the function $h = M_c f$ defined by $h(z) = \sum\limits_{k \in \mathbb{Z}^n_+} c_k a_k z^k$ is in Y. The set of all multipliers from X to Y is denoted by $M_T(X, Y)$. This topic and this definition of coefficient multipliers in analytic spaces the polydisk are not new, previously this problem were considered in papers of M. Nawrocki and M. Gvaradze and others.

We define by $dm_2(\xi)$ the Lebesgue measure on the unit disk and replacing 2 by 2n the Lebesgue measure on the polydisc. Let $L^{\overrightarrow{p}}(\overrightarrow{\alpha})$ be the space of all measurable functions in U^n so that

$$\|f\|_{L^{\overrightarrow{p}}(\overrightarrow{\alpha})} = \Bigg(\int\limits_U (1-|\xi_n|)^{\alpha_n} \Bigg(\int\limits_U (1-|\xi_{n-1}|)^{\alpha_{n-1}} ... \Bigg(\int\limits_U (1-|\xi_1|)^{\alpha_1} |f(\xi_1, ..., \xi_n)|^{p_1} \times$$

$$\times dm_2(\xi_1) \Bigg)^{\frac{p_2}{p_1}} ... dm_2(\xi_{n-1}) \Bigg)^{\frac{p_n}{p_{n-1}}} dm_2(\xi_n) \Bigg)^{\frac{1}{p_n}} < +\infty$$

where all p_j are positive and all $\alpha_j > -1$ for any $j = 1, ..., n$. Let $H^{\overrightarrow{p}}(\overrightarrow{\alpha}) = L^{\overrightarrow{p}}(\overrightarrow{\alpha}) \cap H(U^n)$.

We define now new mixed-norm weighted Hardy analytic spaces in polydisc in the following natural way. Let

$$H^{\overrightarrow{p}}_{\overrightarrow{\alpha}} = \sup_{r_j < 1} \Bigg(\int\limits_T ... \Bigg(\int\limits_T \Bigg(\int\limits_T |f(r\overrightarrow{\xi})|^{p_1} d\xi_1 \Bigg)^{\frac{p_1}{p_2}} d\xi_2 \Bigg)^{\frac{p_3}{p_2}} ... d\xi_n \Bigg)^{\frac{1}{p_n}} \prod_{j=1}^{n} (1-r_j)^{\alpha_j};$$

$p_j \in (0;\infty], \alpha_j \geq 0, j = 1, ..., n.$

Let $F^{q,\infty}_{s,\alpha}$ be a space of analytic functions in the unit polydisk, so that

$$\int\limits_{U^n} \frac{|f(z)|^q (1-|z|)^s (1-|w|)^\alpha}{|1-wz|^{\alpha+1}} dm_{2n}(z)$$

is a bounded function in the unit polydisk as a function of w variable. We assume that $s > -1$ and other values of parameters above are positive numbers.

Below we provide a series of new interesting sharp results on coefficient multipliers in analytic function spaces in the polydisk from recent papers of R. Shamoyan. In the following sharp theorems on multipliers we provide complete description of spaces of coefficient multipliers from $F^{q,\infty}_{s,\alpha}$ to $A^{p,\nu}_{\gamma}$ function spaces and related other pairs of function spaces. We refer for definitions of those several function spaces which appear below in theorems and which can not be seen in this paper to papers of R. Shamoyan which are listed below in references. We refer the readers, for example, to [38],[43],[33],[34], [37], [40]-[41].

In the the following theorem we provide descriptions of spaces of multiplers from $F^{\infty,q}_{sq-1,\alpha}$ to $A^{p,v}_{\gamma}(U^n)$.

Theorem 2.12. [38] *1) Let* $0 < p, v < \infty, \gamma > 0, g \in H(U^n)$.
Let also $g(z) = \sum\limits_{|k|\geq 0}^{\infty} c_{k_1,...,k_n} z^{k_1} ... z^{k_n}$, *be a multiplier from* $F^{\infty,q}_{sq-1,\alpha}$ *to* $A^{p,v}_{\gamma}(U^n)$ *then*

$$\sup_{r\in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1-r_j)^{\gamma+\beta_j+1-s} < \infty$$

for all $\beta_j > s - 1 - \gamma, j = 1, ..., n.$

2) Let $0 < p, v < \infty, \gamma > 0, g \in H(U^n)$. *Let also* $g(z) = \sum\limits_{k_1\geq 0,...,k_n\geq 0}^{\infty} c_{k_1,...,k_n} z^{k_1} ... z^{k_n}$, *be a multiplier from* $F^{\infty,q}_{sq-1,\alpha}$ *to* $F^{p,v}_{\gamma}(U^n)$ *then for* $v \leq p$

$$\sup_{r\in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1-r_j)^{\gamma+\beta_j+1-s} < \infty$$

for all $\beta_j > s - 1 - \gamma, j = 1, ..., n.$

In the following theorem we provide complete descriptions of multipliers from BMOA to$A^{p,q}_{\gamma}(U^n)$ and $F^{p,v}_{\gamma}(U^n)$. We refer for BMOA, $M^{p,q}$ and Y analytic function spaces in the polydisk which can be seen below to corresponding papers which we mentioned below. The topic of study of coefficient multipliers acting between spaces of different dimensions is a new and interesting topic.

Theorem 2.13. [38] *1) Let* $0 < p, v < \infty, \gamma > 0, g \in H(U^n)$.
Let also $g(z) = \sum\limits_{|k|\geq 0}^{\infty} c_{k_1,...,k_n} z^{k_1} ... z^{k_n}$, *be a multiplier from BMOA to* $A^{p,q}_{\gamma}(U^n)$ *then*

$$\sup_{r\in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1-r_j)^{\gamma+\beta_j+2+v} < \infty,$$

where $\beta > -v - 2$ *for any* $v, v > -1$.

2) Let $0 < p, v < \infty, g \in H(U^n)$. *Let also* $g(z) = \sum_{k_1 \geq 0, \ldots, k_n \geq 0}^{\infty} c_{k_1,\ldots,k_n} z^{k_1} \ldots z^{k_n}$, *be a multiplier from BMOA to* $F_\gamma^{p,v}(U^n)$ *then for* $q \leq p$

$$\sup_{r \in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1 - r_j)^{\gamma + \beta_j + 2 + v} < \infty,$$

where $\beta > -v - 2$ *for any* $v, v > -1$.

In the following theorem we provide complete descriptions of multipliers from Y to $F_\gamma^{p,\infty}$ and from Y to $A_\gamma^{p,\infty}$.

Theorem 2.14. [38] *Let* $p, \gamma > 0$. *Let* $g \in H(U^n), g(z) = \sum_{|k| \geq 0} c_k z^k$.

1)If $g \in M_T(Y; F_\gamma^{p,\infty})$ *then*

$$\sup_{r \in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1 - r_j)^{\gamma + \beta_j + 1 - s} < \infty,$$

for all $\beta_j > s - 1 - \gamma, j = 1, \ldots, n$.
2) If $g \in M_T(Y; A_\gamma^{p,\infty})$ *then*

$$\sup_{r \in I^n} M_p(D^\beta g, r) \prod_{j=1}^{n} (1 - r_j)^{\gamma + \beta_j + 1 - s} < \infty,$$

for all $\beta_j > s - 1 - \gamma, j = 1, \ldots, n$.

We assume again

In the following theorems on multipliers we provide description of spaces of coefficient multipliers from $M_\gamma^{p,q}, M_\gamma^{p,\infty}, M_\gamma^{\infty,q}$, to Y function spaces.

Theorem 2.15. [38] *Let* $p, q, \gamma > 0, v > 0$. *Let* $g \in H(U^n), g(z) = \sum_{k \in \mathbb{Z}_+^n} c_k z^k$. *Then*

1) If $g \in (M_T)(M_\gamma^{p,q}, Y)$ *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n - \frac{1}{p}} < \infty,$$

for $\beta > \gamma + \frac{1}{np} - \frac{v}{n} - 1$.
2) If $g \in (M_T)(M_\gamma^{p,\infty}, Y)$ *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n - \frac{1}{p}} < \infty,$$

for $\beta > \gamma + \frac{1}{np} - \frac{v}{n} - 1$.
3) If $g \in (M_T)(M_\gamma^{\infty,q}, Y)$ *then*

$$\sup_{r \in I} M_\infty(D^\beta g, r)(1 - r)^{v + n(\beta+1) - \gamma n} < \infty,$$

for $\beta > \gamma - \frac{v}{n} - 1$.

In the following theorems we provide complete descriptions of multipliers from $F_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$ and $A_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$.

Theorem 2.16. [43] *Let* $g \in H(U^n), g(z) = \sum\limits_{k \in \mathbb{Z}_+^n} c_k z^k$.

Assume $\frac{t}{2} \leq s \leq t < \infty, 0 < p, q \leq 1, 0 < \max(p,q) \leq s \leq 1, t(\alpha + \frac{1}{p}) < 2, m \in \mathbb{N}$ *and* $m > \frac{2}{t} - 1$.

1. $\{c_k\}_{k \in \mathbb{Z}_+^n} \in M_T(F_\alpha^{p,q}(U^n)), A_\beta^{t,s}(U^n))$ *if and only if*

$$\sup_{r \in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

2. $\{c_k\}_{k \in \mathbb{Z}_+^n} \in M_T(A_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n))$ *if and only if*

$$\sup_{r \in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

In the the following theorem we provide descriptions of spaces of multiplers from $F_\alpha^{p,q}(U^n), A_\alpha^{p,q}(U^n)$ to $A_\beta^{t,s}(U^n)$.

Theorem 2.17. [42] *Let* $g \in H(U^n), g(z) = \sum\limits_{k \in \mathbb{Z}_+^n} c_k z^k$.

Assume $t \leq 1, \frac{t}{2} \leq s < t < \infty, 0 < \max(p,q) \leq s < \infty$, $\beta + \frac{1}{t} < \alpha + \frac{1}{t} < \frac{2}{t}, m \in \mathbb{N}$ *and* $m > \frac{2}{t} - 1$.

1.$\{c_k\}_{k \in \mathbb{Z}_+^n} \in M_T(F_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n))$ *if and only if*

$$\sup_{r \in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

2.$\{c_k\}_{k \in \mathbb{Z}_+^n} \in M_T(A_\alpha^{p,q}(U^n), A_\beta^{t,s}(U^n))$ *if and only if*

$$\sup_{r \in I^n} M_t(D^m g, r)(1-r)^{m+1-\frac{1}{p}+\beta-\alpha} < \infty.$$

We considered and concentrated mainly on a problem of description of coefficient multipliers in harmonic spaces of several variables of Bergman type and mixed-norm type, Bloch type however very similar issues related to Hardy and also various weighted Hardy spaces can also be considered in this direction of research. These problems are open and we pose this as a problem to readers. Hardy spaces of harmonic functions in higher dimension were considered in relation with other problems in various papers of A. Aleksandrov, P. Kargaev (see [1] -[3]), T. Wolff [48] and others in last several decades. Various interesting new sharp results concerning estimates of Taylor coefficients, the action of fractional integration operator and differentiation operators and related issues can be seen in recent papers of the E. Rodikova [30],[31], E. Shubabko and their coauthors and also in papers of R. Mestrovic, V. Gavrilov and their coauthors.

Similar results on coefficient multipliers based on similar ideas probably are valid also in harmonic function spaces in the upper half-plane namely in $\mathbb{R}_+^{n+1}$. We leave this interesting open problem to interested readers. We refer the reader to research papers [33]-[19] of R. Shamoyan, these are various research papers of R. Shamoyan related to various problems in harmonic function spaces of several variables.

New rather interesting mixed-norm Bergman type analytic function spaces were considered by O. Yaroslavceva in [49] in the unit polydisk and certain new spaces of coefficient multipliers were described completely in that paper for these mixed-norm function spaces. We refer the interested reader to [49] for these sharp results. Issues related to fractional integration and differentiation and to Taylor coefficients are closely connected with the study of coefficient

multipliers, not only in analytic but also in harmonic function spaces of several variables, and many interesting open problems arise in this direction (see, for example, [20] and the references therein).

We didn't discuss and consider in this note various aspects of this topic related to coefficient multipliers in spaces of analytic functions in the ball or in various other complicated domains in higher dimension, though we can consider and define spaces of coefficient multipliers in various way in such type domains also using same type taylor expansion of analytic functions. As far as we know, these issues and problems in complex function theory are novel. These new issues may be interesting for various interested readers in the future. We pose this as a problem.

3. Conclusion

Similar problems and constructions related to this topic of coefficient multipliers in various function spaces may be considered or provided also in various other analytic and harmonic function spaces not only in the polydidk and in the unit ball of $\mathbb{R}^n$. Here first of all appropriate definitions of coefficient multipliers must be given based on Taylor expansions of analytic functions in holomorphic spaces in these domains.

Another interesting and maybe more difficult problem which we may suggest to readers is related to similar type function spaces but in product domains, here appropriate natural extensions must be first provided based on analytic and harmonic spaces which we considered in this note in one domain and then natural extensions of definitions of coefficient multipliers should be given for such type spaces.

We didn't discuss in this note certain new direct generalizations of the unit polydisk so called polyballs (products of unit balls in C^n) and all results of this note obtained in the polydisk may be probably extended to the case of polyballs. Many interesting new sharp results in this rather general new domain in C^n were obtained recently by the first author.

References

[1] A. B. Aleksandrov and P. Kargaev, St. Petersburg Math. J. **5** (1994), no. 2, 229–286; translated from Algebra i Analiz **5** (1993), no. 2, 1–73.
[2] A. B. Aleksandrov, Essays on nonlocally convex Hardy classes, in *Complex analysis and spectral theory (Leningrad, 1979/1980)*, pp. 1–89, Lecture Notes in Math., 864, Springer, Berlin.
[3] A. B. Aleksandrov and P. Kargaev, St. Petersburg Math. J. **5** (1994), no. 2, 229–286; translated from Algebra i Analiz **5** (1993), no. 2, 1–73.
[4] M. Arsenovic and R. F. Shamoyan, Sharp theorems on multipliers in harmonic function spaces in higher dimension, Acta Comment. Univ. Tartu. Math. **16** (2012), no. 2, 167–180.
[5] M. Arsenovic and R. F. Shamoyan, Sharp theorems on multipliers in harmonic function spaces in higher dimension, Acta Comment. Univ. Tartu. Math. **16** (2012), no. 2, 167–180.
[6] M. Arsenovic and R. F. Shamoyan, On embeddings, traces and distances in harmonic function spaces, C. R. Acad. Bulgare Sci. **65** (2012), no. 10, 1319–1324.
[7] M. Arsenovic and R. F. Shamoyan, On embeddings, traces and multipliers in harmonic function spaces, Kragujevac J. Math. **37** (2013), no. 1, 45–64.
[8] M. Arsenovic and R. F. Shamoyan, On some new theorems on multipliers in harmonic function spaces in higher dimension II, Bull. Korean Math. Soc. **50** (2013), no. 5, 1451–1469.
[9] M. Arsenovic and R. F. Shamoyan, Trace theorems in harmonic function spaces on $(\mathbb{R}^{n+1}_+)^m$, multipliers theorems and related problems, Kragujevac J. Math. **35** (2011), no. 3, 411–430.

[10] O. Blasco, Multipliers on spaces of analytic functions, Canad. J. Math. **47** (1995), no. 1, 44–64.
[11] O. Blasco and J. L. Arregui Casaus, Multipliers on vector valued Bergman spaces, Canad. J. Math. **54** (2002), no. 6, 1165–1186.
[12] J. Bourgain and T. H. Wolff, A remark on gradients of harmonic functions in dimension ≥ 3, Colloq. Math. **60/61** (1990), no. 1, 253–260.
[13] A. P. Calderón, On the behaviour of harmonic functions at the boundary, Trans. Amer. Math. Soc. **68** (1950), 47–54.
[14] A. P. Calderón and A. S. Zygmund, On higher gradients of harmonic functions, Studia Math. **24** (1964), 211–226.
[15] L. A. E. Carleson, On the existence of boundary values for harmonic functions in several variables, Ark. Mat. **4** (1962), 393–399 (1962).
[16] R. R. Coifman and R. Rochberg, Representation theorems for holomorphic and harmonic functions in L^p, in *Representation theorems for Hardy spaces*, pp. 11–66, Astérisque, 77, Soc. Math. France, Paris.
[17] R. R. Coifman and G. L. Weiss, Representations of compact groups and spherical harmonics, Enseign. Math. (2) **14** (1968), 121–173 (1969).
[18] P. L. Duren and A. L. Shields, Coefficient multipliers of H^p and B^p spaces, Pacific J. Math. **32** (1970), 69–78.
[19] A. È. Dzhrbashyan and F. A. Shamoyan, *Topics in the theory of A^p_α spaces*, Teubner-Texte zur Mathematik, 105, Teubner, Leipzig, 1988.
[20] K. S. Eikrem and E. Malinnikova, Coefficients multipliers of weighted spaces of harmonic functions, Integral Equations Operator Theory **82** (2015), no. 4, 555–573.
[21] R. A. Hunt and R. L. Wheeden, On the boundary values of harmonic functions, Trans. Amer. Math. Soc. **132** (1968), 307–322.
[22] M. Jevtić and M. Pavlović, Coefficient multipliers on spaces of analytic functions, Acta Sci. Math. (Szeged) **64** (1998), no. 3-4, 531–545.
[23] M. Jevtić, D. Vukotić and M. Arsenovic, *Taylor coefficients and coefficient multipliers of Hardy and Bergman-type spaces*, RSME Springer Series, 2, Springer, Cham, 2016.
[24] M. S. Mateljević and M. Pavlović, Multipliers of H^p and BMOA, Pacific J. Math. **146** (1990), no. 1, 71–84.
[25] M. Nawrocki, Multipliers, linear functionals and the Fréchet envelope of the Smirnov class $N_*(\mathbf{U}^n)$, Trans. Amer. Math. Soc. **322** (1990), no. 2, 493–506.
[26] V. Oleinik, *Embedding theorems for weighted classes of harmonic and analytic functions*, Journal of Soviet Mathematics, 9, (1978) 228–243.
[27] V. L. Oleinik, Imbedding theorems for weighted classes of harmonic and analytic functions, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **47** (1974), 120–137, 187, 192–193.
[28] J. M. Ortega and J. Fàbrega, Pointwise multipliers and corona type decomposition in BMOA, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 1, 111–137.
[29] M. Pavlović, Multipliers of the vanishing Hardy classes, Publ. Inst. Math. (Beograd) (N.S.) **52(66)** (1992), 34–36.
[30] E. G. Rodikova, Coefficient multipliers for the Privalov class in a disk, Zh. Sib. Fed. Univ. Mat. Fiz. **11** (2018), no. 6, 723–732.
[31] E. G. Rodikova, On coefficient multipliers for area Privalov classes, Ufa Math. J. **13** (2021), no. 4, 80–90.
[32] R. F. Shamoyan and A. Abkar, On multipliers of spaces of harmonic functions in the unit ball of $\mathbb{R}^n$, J. Inequal. Spec. Funct. **3** (2012), no. 1, 1–9.
[33] R. F. Shamoyan, Ukrainian Math. J. **52** (2000), no. 10, 1606–1617; translated from Ukraïn. Mat. Zh. **52** (2000), no. 10, 1405–1414.
[34] R. F. Shamoyan, Russian Math. (Iz. VUZ) **44** (2000), no. 7, 63–65; translated from Izv. Vyssh. Uchebn. Zaved. Mat. **2000**, no. 7, 67–69.
[35] R. F. Shamoyan, Izv. Nats. Akad. Nauk Armenii Mat. **34** (1999), no. 4J. Contemp. Math. Anal. **34** (1999), no. 4.

[36] R. F. Shamoyan, *Bounded projections and descriptions of multipliers in harmonic function spaces in unit ball*, Proc. Math. Center of Lobachevski, Kazan, (1998) 260–261 (in Russian).

[37] R. F. Shamoyan, Siberian Math. J. **43** (2002), no. 1, 169–182; translated from Sibirsk. Mat. Zh. **43** (2002), no. 1, 212–227, iv.

[38] R. Shamoyan, *On multipliers of some new analytic function spaces in polydisc of BMOA type*, Mathematical notes of NEFU., 33(1), (2026) 73–91.

[39] S. M. Kurilenko and R. F. Shamoyan, On Bergman type integral operators in analytic spaces in tubular domains over symmetric cones, Kragujevac J. Math. **41** (2017), no. 2, 295–312.

[40] R. F. Shamoyan and S. M. Kurilenko, On multipliers of some new analytic Hardy-type $H^{\vec{P}}_{\vec{\alpha}}$ and Bergman-type $H^{\vec{P}}(\vec{\alpha})$ function spaces in polydisc, ROMAI J. **10** (2014), no. 1, 187–204.

[41] R. F. Shamoyan, *On multipliers of some new analytic spaces in bounded domains in* $\mathbb{C}^n$, ROMAI Journal, 21(1), (2025) 111–124.

[42] R. F. Shamoyan, M. Arsenović, *On multipliers of holomorphic* $F^{p,q}_{\alpha}$ *type spaces on the unit polydisc*, Journ. of Siberian Federal University. Mathematics and Physics, 5(4), (2012) 471–479.

[43] R. F. Shamoyan and O. R. Mihić, On multipliers of some new analytic $M^{p,q}_{\alpha}$, $M^{p,\infty,\alpha}$ and $M^{\infty,p,\alpha}$ type spaces and related spaces on the unit polydisc, Math. Montisnigri **27** (2013), 9–37.

[44] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of harmonic functions, J. Reine Angew. Math. **299/300** (1978), 256–279.

[45] E. M. Stein, On the theory of harmonic functions of several variables. II. Behavior near the boundary, Acta Math. **106** (1961), 137–174.

[46] E. M. Stein and G. L. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series, No. 32, Princeton Univ. Press, Princeton, NJ, 1971.

[47] E. M. Stein and G. L. Weiss, On the theory of harmonic functions of several variables. I. The theory of H^p-spaces, Acta Math. **103** (1960), 25–62.

[48] T. H. Wolff, Counterexamples with harmonic gradients in $\mathbf{R}^3$, in *Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991)*, 321–384, Princeton Math. Ser., 42, Princeton Univ. Press, Princeton, NJ.

[49] O. V. Yaroslavtseva, J. Math. Sci. (New York) **107** (2001), no. 4, 4143–4146; translated from Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **255** (1998), 244–248, 255.

©Poincare Publishers, India