Poincare Journal of Mathematical Physics and Applications
Vol. 1, No. 2 (2026), 135-148
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DOI: 10.46753/pjmpa.2026.v01i02.004
Online Published on 27.11.2026

Bases in Hilbert Spaces: Structure and Classification

Raksha Sharma[1,†]

[1]Department of Physics, Kirori Mal College, University of Delhi, Delhi-110007.
rakshakmc@gmail.com

Submitted: 29.10.2026 | Revised: 22.11.2026 | Accepted: 25.11.2026

Abstract. We survey several notions of basis in real and complex Hilbert spaces: Hamel, orthogonal and orthonormal, Schauder, normalized and semi-normalized Schauder, unconditional and conditional, Riesz, Markushevich, and biorthogonal systems. We also discuss Hilbertian and Besselian basic sequences, frame-related systems, and several notions of minimality and completeness. Particular attention is given to the fact that normalization alone does not turn a Schauder basis into a Riesz basis. An explicit normalized conditional Schauder basis is obtained from a weighted Fourier system. Characterization theorems, examples, and a diagram of logical implications clarify the distinctions. The presentation is expository and attributes the classical results to the relevant literature.

Keywords: Hilbert space; Schauder basis; Riesz basis; unconditional basis; orthonormal basis; biorthogonal system; Markushevich basis; frame.
MSC (2020): 46B15, 46C05, 42C15, 42C30
Communicated by: Amit Kumar
[†]**Corresponding author**

1. Introduction and Preliminaries

The word "basis" is used in several distinct senses in functional analysis. In finite-dimensional Hilbert spaces, every algebraic basis is a Schauder basis, and every basis is the image of an orthonormal basis under an invertible operator. In infinite dimensions these statements require considerable qualification. Algebraic spanning, norm-convergent expansion, unconditional convergence, and uniform two-sided coefficient estimates are different properties.

Throughout, $\mathcal{H}$ is a real or complex Hilbert space, usually separable and infinite-dimensional, and inner products are linear in the first variable. A sequence is indexed by $\mathbb{N}$ unless otherwise indicated. Every topological basis considered below is complete in its ambient space; a basic sequence means a Schauder basis of its closed linear span. Our primary concern is with countable systems and norm convergence. For background, see [8, 10, 2, 5, 9].

Definition 1.1. A sequence (x_n) is *complete* (or *total*) if $\overline{\operatorname{span}}\{x_n : n \geq 1\} = \mathcal{H}$. It is *minimal* if $x_n \notin \overline{\operatorname{span}}\{x_k : k \neq n\}$ for each n. It is *uniformly minimal* if

$$\inf_n \operatorname{dist}\big(x_n/\|x_n\|, \overline{\operatorname{span}}\{x_k : k \neq n\}\big) > 0.$$

It is *semi-normalized* if $0 < \inf_n \|x_n\| \leq \sup_n \|x_n\| < \infty$, and *normalized* if $\|x_n\| = 1$ for every n.

Definition 1.2. Sequences (x_n) and (y_n) are *biorthogonal* if $\langle x_n, y_m\rangle = \delta_{nm}$ for all m, n.

Proposition 1.3. Every minimal sequence (x_n) admits a unique biorthogonal sequence (y_n) in $M = \overline{\operatorname{span}}\{x_n\}$. In particular, a complete minimal sequence has a unique biorthogonal sequence in $\mathcal{H}$.

Proof. Fix n and let $M_n = \overline{\operatorname{span}}\{x_k : k \neq n\}$. Minimality gives $x_n \notin M_n$. Write $x_n = u_n + v_n$ with $u_n \in M_n$ and $0 \neq v_n \perp M_n$. Then $y_n = v_n/\|v_n\|^2$ satisfies $\langle x_k, y_n\rangle = \delta_{kn}$. If two vectors in M have these inner products, their difference is perpendicular to the dense subspace $\operatorname{span}\{x_k\}$ of M, so they coincide. □

Remark 1.4. Completeness and minimality do not, by themselves, ensure that every vector has a convergent series expansion in the system. A complete minimal system is sometimes called an *exact system*; this is weaker than being a Schauder basis.

2. Hamel, orthogonal, and orthonormal bases

Finite algebraic expansions and infinite orthogonal expansions answer different representation questions. The distinction is particularly sharp in infinite dimension, where the Baire category theorem rules out a countable Hamel basis. Orthogonal expansions, in contrast, are governed by the Pythagorean identity.

Definition 2.1. A *Hamel basis* is an algebraic basis: each $x \in \mathcal{H}$ is a *finite* linear combination of its elements, uniquely. An *orthogonal basis* is a complete family of nonzero pairwise orthogonal vectors. An *orthonormal basis* (ONB) is a complete orthonormal family.

The significance of an orthonormal basis lies in its ability to provide a canonical representation of every element of a Hilbert space through its Fourier coefficients. Moreover, orthonormality ensures that the norm of an element is completely determined by these coefficients. The following theorem establishes these fundamental properties, which form the foundation for the study of more general basis systems in Hilbert spaces.

Theorem 2.2 (Fourier expansion and Parseval)**.** *If (e_n) is an orthonormal basis of a separable Hilbert space, then every $x \in \mathcal{H}$ satisfies*

$$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \qquad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2.$$

The series converges unconditionally in norm.

Proof. Let $P_N x = \sum_{n=1}^{N} \langle x, e_n\rangle e_n$. Orthogonality yields $\|x - P_N x\|^2 = \|x\|^2 - \sum_{n=1}^{N} |\langle x, e_n\rangle|^2$. Bessel's inequality shows the coefficients belong to ℓ^2, so the partial sums converge to some y. Then $x - y$ is perpendicular to every e_n and hence zero by completeness. Parseval follows by taking limits. Orthogonality makes every rearranged partial sum a Cauchy net, giving unconditional convergence. □

Orthogonality provides a natural framework for obtaining unique expansions in Hilbert spaces, even when the underlying vectors are not normalized. While completeness and orthogonality of a sequence of nonzero vectors are sufficient to guarantee the Schauder basis property, the Riesz basis property requires additional control over the norms of the basis elements. The following proposition makes this distinction precise.

Proposition 2.3. A complete orthogonal sequence (u_n) is a Schauder basis, with coefficients $\langle x, u_n\rangle/\|u_n\|^2$. It is a Riesz basis if and only if

$$0 < \inf_n \|u_n\| \le \sup_n \|u_n\| < \infty.$$

Proof. Normalize $e_n = u_n/\|u_n\|$ and apply the preceding theorem. Moreover, $\|\sum_{n=1}^N a_n u_n\|^2 = \sum_{n=1}^N |a_n|^2\|u_n\|^2$, which gives both directions of the Riesz inequalities. □

Remark 2.4. A countably infinite Hamel basis cannot exist in an infinite-dimensional Banach space. Indeed, if $\mathcal{H} = \bigcup_{N\ge1} \operatorname{span}\{h_1, \dots, h_N\}$, the Baire category theorem would force one finite-dimensional (hence closed, nowhere-dense) proper subspace to have nonempty interior, a contradiction. Thus an infinite-dimensional separable Hilbert space has a countable orthonormal basis but no countable Hamel basis.

3. Schauder bases and basis projections

Schauder bases extend the notion of orthonormal bases by retaining the essential property of unique norm-convergent expansions without imposing orthogonality. The associated basis projections provide a natural means of studying the convergence of these expansions through finite-dimensional approximations. Their uniform boundedness plays a fundamental role in understanding the structural and stability properties of Schauder bases. This section examines these concepts and establishes several basic results that are instrumental in the subsequent study of Riesz and unconditional bases.

The defining feature of a Schauder basis is convergence in a prescribed order. Its coordinate projections translate this qualitative requirement into a uniform operator estimate; see [8, 5].

Definition 3.1. A sequence (x_n) is a *Schauder basis* for $\mathcal{H}$ if every $x \in \mathcal{H}$ has a unique norm-convergent representation

$$x = \sum_{n=1}^{\infty} a_n x_n.$$

Its coefficient functionals are $x_n^*(x) = a_n$, and its partial-sum projections are $P_N x = \sum_{n=1}^N x_n^*(x) x_n$.

A fundamental problem in the theory of Schauder bases is to characterize those sequences that admit unique norm-convergent expansions. The behavior of finite partial sums provides an effective criterion for this purpose. In particular, the uniform control of initial partial sums by the corresponding full finite sums, together with completeness, characterizes Schauder bases. The following theorem establishes this characterization and introduces the basis constant as a quantitative measure of such control.

Theorem 3.2 (Basis projection criterion)**.** *A sequence (x_n) of nonzero vectors is a Schauder basis for $\mathcal{H}$ if and only if it is complete and there exists $K < \infty$ such that, for all scalars $a_1, \dots, a_m$ and $1 \le N \le m$,*

$$\left\| \sum_{n=1}^{N} a_n x_n \right\| \le K \left\| \sum_{n=1}^{m} a_n x_n \right\|. \tag{3.1}$$

The least admissible K is the basis constant*.*

Proof. For a Schauder basis, each P_N is bounded: the coefficient functionals are continuous by the standard closed-graph argument applied to the coefficient map, or by the classical basis theorem. Since $P_N x \to x$ for every x, uniform boundedness gives $\sup_N \|P_N\| < \infty$, which implies (3.1).

Conversely, if a finite combination vanishes, (3.1) forces every initial sum to vanish. Taking differences and using $x_n \neq 0$ proves linear independence. For each N, the truncation map on finite linear combinations is bounded by K and therefore extends uniquely to a bounded operator P_N on $\mathcal{H}$. Completeness and (3.1) show $P_N x \to x$, first on finite linear combinations and then by density. The identities $P_N P_M = P_{\min(N,M)}$ show that the successive differences have one-dimensional ranges, so $P_N - P_{N-1}$ (with $P_0 = 0$) gives the required scalar multiples of x_N. Applying each bounded coordinate projection to any convergent expansion establishes uniqueness. □

Remark 3.3. A Schauder basis is not required to be normalized or unconditional. Multiplying each basis vector by any nonzero scalar preserves the Schauder-basis property, since the partial-sum projections remain unchanged; it need not preserve the Riesz-basis property.

4. Riesz bases

A Riesz basis transports the geometry of an orthonormal basis through an isomorphism. The resulting coefficient estimates are independent of the length of a finite expansion [10, 2].

Definition 4.1. A sequence (x_n) is a *Riesz basis* for $\mathcal{H}$ if it is the image of an orthonormal basis (e_n) under a bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$, that is, $x_n = Te_n$.

The distinguishing feature of a Riesz basis is the quantitative relationship between the norm of a finite linear combination of basis elements and the Euclidean norm of its coefficients. This relationship ensures stability of representation and prevents excessive amplification or attenuation of coefficient sequences. The following theorem provides a fundamental characterization of Riesz bases in terms of two-sided norm inequalities.

Theorem 4.2 (Riesz-basis inequalities)**.** *A sequence (x_n) is a Riesz basis if and only if it is complete and there exist $0 < A \le B < \infty$ such that for all finite scalar sequences (a_n),*

$$A \sum_n |a_n|^2 \le \left\| \sum_n a_n x_n \right\|^2 \le B \sum_n |a_n|^2. \tag{4.1}$$

Proof. If $x_n = Te_n$, then (4.1) holds with $A = \|T^{-1}\|^{-2}$ and $B = \|T\|^2$. Conversely, define T on finite linear combinations of the e_n by $T(\sum a_n e_n) = \sum a_n x_n$. The upper estimate extends T boundedly to $\mathcal{H}$. The lower estimate implies that T is injective with closed range

and has a bounded inverse on that range. Completeness makes its range dense and hence equal to $\mathcal{H}$. □

Corollary 4.3. Every Riesz basis is a semi-normalized Schauder basis, and every Riesz basis is unconditional.

Proof. The operator representation transfers the unconditional orthonormal expansion to the Riesz basis. Taking only one nonzero coefficient in (4.1) gives $\sqrt{A} \le \|x_n\| \le \sqrt{B}$. □

Example 4.4 (A Schauder basis that is not Riesz)**.** In ℓ^2, set $x_n = e_n/n$. For $x = \sum a_n e_n$, $\sum_{n=1}^{N} (na_n)x_n = \sum_{n=1}^{N} a_n e_n \to x$, uniquely. Thus (x_n) is an orthogonal Schauder basis. But $\|x_n\| = 1/n \to 0$, contradicting the uniform lower bound for a Riesz basis.

Example 4.5 (A nonorthogonal Riesz basis)**.** Let S be the unilateral right shift on ℓ^2, let $|\alpha| < 1$, and set $T = I + \alpha S$. The Neumann series makes T invertible. Hence $x_n = e_n + \alpha e_{n+1}$ is a Riesz basis, with bounds $(1 - |\alpha|)^2$ and $(1 + |\alpha|)^2$.

5. Unconditional and conditional Schauder bases

An important aspect of Schauder basis theory is the dependence of convergence on the ordering of the basis elements. When the expansion of every element remains convergent under arbitrary rearrangements, the basis is termed unconditional; otherwise, it is called conditional. This distinction reveals significant differences in the geometric and approximation properties of basis systems. In Hilbert spaces, unconditionality is particularly important because of its close relationship with the Riesz basis property. The present section explores these notions and their fundamental characterizations. Changing the order of an expansion can affect its convergence. In Hilbert spaces, averaging over sign changes connects order-independent convergence to square-summable coefficients [5, 9].

Definition 5.1. A Schauder basis (x_n) is *unconditional* if each basis expansion converges in norm after every permutation of its terms. Otherwise it is *conditional*.

Unconditionality of a Schauder basis reflects the stability of its expansions under changes in the ordering of the terms. This qualitative convergence property can be expressed quantitatively through the uniform boundedness of finite sign-changing transformations. The following theorem establishes this equivalence and provides a useful analytical tool for investigating unconditional bases in both real and complex Hilbert spaces.

Theorem 5.2 (Unconditionality criterion)**.** *A Schauder basis (x_n) is unconditional if and only if there exists $C < \infty$ such that for all finite scalar sequences (a_n) and all signs $\varepsilon_n \in \{-1, 1\}$,*

$$\left\| \sum_n \varepsilon_n a_n x_n \right\| \le C \left\| \sum_n a_n x_n \right\|. \tag{5.1}$$

For complex scalars, signs may equivalently be replaced by unimodular multipliers, with possibly different constants.

Proof. If the basis is unconditional, every coordinate suppression projection $P_F(\sum a_n x_n) = \sum_{n \in F} a_n x_n$ is defined and bounded. For each x, the unconditional expansion makes $\sup_F \|P_F x\| < \infty$; the uniform boundedness principle yields $M = \sup_F \|P_F\| < \infty$. A

sign-change operator on finite sums is $2P_F - I$, so (5.1) holds with $C \le 2M + 1$. Conversely, (5.1) implies uniformly bounded suppression operators, because $P_F = (I + D_\varepsilon)/2$ on finite sums. Approximation by a sufficiently long initial partial sum then proves that all finite tails are uniformly small, which is the unconditional Cauchy criterion. For complex multipliers, write each multiplier as its real and imaginary parts. A finite real multiplier vector in $[-1, 1]^N$ is a convex combination of sign vectors; the triangle inequality therefore gives the complex multiplier estimate with constant $2C$. □

In Hilbert spaces, the interplay between unconditional convergence and the geometric structure of the underlying space leads to a particularly elegant characterization of Riesz bases. Although unconditionality alone does not guarantee the Riesz basis property, an additional uniform control on the norms of the basis elements is sufficient. The following theorem establishes this fundamental connection and highlights the role of semi-normalization in the classification of Schauder bases.

Theorem 5.3 (Hilbert-space characterization)**.** *For a Schauder basis (x_n) of a Hilbert space, the following are equivalent:*

(i) (x_n) *is a Riesz basis;*
(ii) (x_n) *is unconditional and semi-normalized;*
(iii) (x_n) *is unconditional and normalized up to rescaling by scalars bounded above and away from zero in modulus.*

In particular, every normalized unconditional Schauder basis in a Hilbert space is a Riesz basis.

Proof. The first assertion implies the second by the preceding results. Assume (ii), and let C be as in (5.1). For independent Rademacher signs ε_n, orthogonality in expectation gives

$$\mathbb{E}\left\|\sum_{n=1}^{N} \varepsilon_n a_n x_n\right\|^2 = \sum_{n=1}^{N} |a_n|^2 \|x_n\|^2.$$

Applying (5.1) first to (a_n) and then to $(\varepsilon_n a_n)$ yields

$$C^{-2}\left\|\sum a_n x_n\right\|^2 \le \sum |a_n|^2 \|x_n\|^2 \le C^2\left\|\sum a_n x_n\right\|^2.$$

If $m \le \|x_n\| \le M$, this becomes

$$\frac{m^2}{C^2}\sum |a_n|^2 \le \left\|\sum a_n x_n\right\|^2 \le C^2 M^2 \sum |a_n|^2.$$

Completeness and the Riesz-basis inequalities prove (i). The equivalence of (ii) and (iii) follows from the definition of semi-normalization. □

Remark 5.4. Unconditionality *without* semi-normalization is insufficient: the orthogonal basis (e_n/n) is unconditional but not Riesz. Conversely, normalization *without* unconditionality is insufficient, as the next section demonstrates.

6. A normalized Schauder basis which is not Riesz

Although every Riesz basis in a Hilbert space is a Schauder basis, the converse is not true, even when the Schauder basis is normalized. Indeed, normalization ensures that

all basis vectors have unit norm, but it does not guarantee the uniform upper and lower estimates required for the Riesz basis property. This distinction highlights the essential role of unconditionality in the characterization of Riesz bases. In this section, we present a counterexample demonstrating that a normalized Schauder basis in a Hilbert space need not be a Riesz basis, thereby illustrating the fundamental difference between these two classes of bases. We give an explicit example, using a classical weighted Fourier-series theorem. Identify the circle with $(-\pi, \pi]$ and fix $0 < \alpha < 1$. Put $w(t) = |t|^\alpha$ (its value at $t = 0$ is irrelevant) and

$$\mathcal{H}_w = L^2((-\pi, \pi), w(t)\,dt), \qquad c = \left(\int_{-\pi}^{\pi} w(t)\,dt\right)^{-1/2}.$$

Enumerate the integers in symmetric order $0, 1, -1, 2, -2, \ldots$ and let

$$x_1(t) = c, \quad x_{2k}(t) = ce^{ikt}, \quad x_{2k+1}(t) = ce^{-ikt} \quad (k \geq 1). \tag{6.1}$$

Every x_n has norm one. The preceding construction provides an explicit setting in which the Schauder basis property and normalization hold without the stronger stability conditions required for a Riesz basis. In particular, the following theorem shows that normalization alone does not ensure the Riesz basis property in a Hilbert space.

Theorem 6.1. *The sequence* (6.1) *is a normalized Schauder basis of $\mathcal{H}_w$, but it is not a Riesz basis.*

Proof. The power weight $w(t) = |t|^\alpha$, $-1 < \alpha < 1$, belongs to the Muckenhoupt A_2 class on the circle [7]: indeed, averages of w and w^{-1} over intervals have uniformly bounded product. By the weighted conjugate-function theorem of Hunt, Muckenhoupt and Wheeden [6] (and its Fourier partial-sum consequence), the symmetric Fourier partial-sum operators

$$S_N f = \sum_{|k| \leq N} \widehat{f}(k) e^{ikt}, \qquad \widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\,dt,$$

are uniformly bounded on $L^2(w)$. The Fourier coefficient functionals are continuous there because $w^{-1} \in L^1$, and trigonometric polynomials are dense. Thus the symmetric partial sums converge to f in $L^2(w)$. In the ordering (6.1), the remaining intermediate partial sum adds only one Fourier mode; its operator norm is uniformly bounded because each coefficient functional for the unnormalized exponentials has the same norm and the exponentials have the same $L^2(w)$ norm. Hence the basis projection criterion proves that (6.1) is a Schauder basis.

Suppose it were a Riesz basis. Then there would be constants $A, B > 0$ such that, for every trigonometric polynomial p,

$$A \sum_k |a_k|^2 \leq c^2 \int_{-\pi}^{\pi} |p(t)|^2 w(t)\,dt \leq B \sum_k |a_k|^2, \quad p(t) = \sum_k a_k e^{ikt}.$$

Parseval gives $\sum |a_k|^2 = (2\pi)^{-1} \int |p|^2$. Thus the weighted and unweighted L^2 norms would be uniformly equivalent on trigonometric polynomials. Since w is bounded above, approximation in ordinary L^2 also gives approximation in $L^2(w)$. The lower estimate consequently holds for every ordinary L^2 function. Applying it to the indicator of $(-\varepsilon, \varepsilon)$ gives $A/(2\pi) \leq c^2 \varepsilon^\alpha$, which is impossible as $\varepsilon \downarrow 0$. The basis is therefore conditional by the preceding characterization. □

Remark 6.2. For a real Hilbert-space version, use the real system $1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots$, normalizing each vector in $L^2(w)$. The cosine and sine norms are bounded above and away

from zero: their squared norms tend to $\frac{1}{2}\int w$ by the Riemann–Lebesgue lemma and each is positive. The same partial-sum argument and the indicator-function contradiction apply, using ordinary real Fourier Parseval. The nontrivial input is the classical weighted Fourier theorem; the example is not claimed to be elementary. The commonly suggested near-parallel two-vector blocks do *not* give a Schauder basis when their within-block projection norms diverge. This distinction is essential when constructing valid counterexamples.

7. Biorthogonal bases and dual expansions

Biorthogonality provides a natural extension of orthogonality by establishing a dual relationship between two sequences in a Hilbert space. In the absence of orthogonality, biorthogonal systems offer an effective mechanism for identifying expansion coefficients and recovering elements through dual representations. The existence and properties of biorthogonal sequences are closely connected with minimality, completeness, and the uniqueness of basis expansions. This section explores these relationships and emphasizes the role of dual systems in the structural analysis of Hilbert space bases. Coordinate recovery can be expressed through inner products with a second sequence. For a Riesz basis this sequence is obtained by applying the inverse adjoint of the generating operator.

The usefulness of biorthogonal sequences becomes particularly evident in the representation of elements with respect to nonorthogonal bases. For a Schauder basis, the associated biorthogonal sequence determines the coefficients of its unique expansion. In the case of a Riesz basis, the relationship is stronger, since the biorthogonal sequence itself forms a Riesz basis, allowing the roles of the original and dual sequences to be interchanged in the reconstruction formula. The following proposition establishes these representation properties.

Proposition 7.1. Let (x_n) be a Schauder basis of $\mathcal{H}$ and (y_n) its biorthogonal sequence. Then

$$x = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n \qquad (x \in \mathcal{H}).$$

If (x_n) is a Riesz basis with $x_n = Te_n$, then $y_n = (T^{-1})^* e_n$ is also a Riesz basis and

$$x = \sum_{n} \langle x, y_n \rangle x_n = \sum_{n} \langle x, x_n \rangle y_n.$$

Proof. The first formula is the defining Schauder expansion, with the coefficient functionals represented by Riesz's representation theorem. In the Riesz case, $\langle Te_n, (T^{-1})^* e_m \rangle = \langle e_n, e_m \rangle = \delta_{nm}$. Since $(T^{-1})^*$ is bounded and invertible, (y_n) is a Riesz basis. Applying the first expansion to each of the two dual bases proves both identities. □

Remark 7.2. The biorthogonal sequence of a general Schauder basis need not itself be a Schauder basis for the whole dual in an arbitrary Banach space. Hilbert spaces are reflexive, however, and the biorthogonal sequence of a Schauder basis of a Hilbert space is a Schauder basis of the Hilbert space under the Riesz identification of $\mathcal{H}$ with its dual. This is the shrinking-basis property of reflexive spaces.

8. Markushevich bases and exact systems

Markushevich bases provide a broader framework for the study of basis systems by relaxing the requirement of norm-convergent expansions while preserving completeness and biorthogonality.

The notion of exactness, expressed through completeness and minimality, captures the essential nonredundancy of a system. However, a Markushevich basis additionally requires the completeness of its associated biorthogonal sequence. These concepts reveal important distinctions between algebraic independence, topological completeness, and representation properties in Hilbert spaces. This section examines their fundamental characteristics and relationships with Schauder and Riesz bases. A biorthogonal pair may separate points without furnishing convergent ordered expansions. This observation motivates the following intermediate notions [8, 5].

Definition 8.1. A *Markushevich basis* (or *M-basis*) is a biorthogonal pair (x_n, y_n) for which both $\operatorname{span}\{x_n\}$ and $\operatorname{span}\{y_n\}$ are dense in $\mathcal{H}$. An M-basis is *strong* if for every $x \in \mathcal{H}$,

$$x \in \overline{\operatorname{span}}\{x_n : \langle x, y_n\rangle \neq 0\}.$$

Although Markushevich bases do not generally guarantee norm-convergent series representations, the additional structure of a Schauder basis yields a stronger approximation property. In particular, each element belongs to the closed linear span of those basis vectors for which its corresponding biorthogonal coefficients are nonzero. This observation establishes the following connection between Schauder bases and strong Markushevich bases.

Proposition 8.2. Every Schauder basis of a Hilbert space, together with its biorthogonal sequence, is a strong Markushevich basis.

Proof. The basis vectors are complete. If z is orthogonal to every y_n, then all Schauder coefficients of z vanish, hence $z = 0$; thus (y_n) is complete. Finally, the Schauder expansion of x uses only indices for which $\langle x, y_n\rangle \neq 0$, establishing the strong property. □

Remark 8.3. The converse need not hold: biorthogonality, totality of both systems, and even the strong M-basis property do not by themselves yield uniformly bounded ordered partial-sum projections. An M-basis is therefore a weaker concept than a Schauder basis.

9. Hilbertian and Besselian systems

Hilbertian and Besselian systems arise naturally in the study of the relationship between scalar coefficients and their associated vector expansions in Hilbert spaces. These notions capture two complementary aspects of the Riesz basis inequalities, namely, the upper and lower control of the norms of finite linear combinations. Their study provides a deeper understanding of the convergence, stability, and coefficient structure of basis systems. This section examines their fundamental properties and establishes their relationship with Riesz bases. Upper and lower synthesis estimates address complementary questions: whether square-summable coefficients produce convergent series and whether convergent expansions have square-summable coefficients.

Definition 9.1. A sequence (x_n) is *Hilbertian* if there is $B > 0$ such that

$$\left\| \sum_n a_n x_n \right\|^2 \leq B \sum_n |a_n|^2$$

for all finitely supported scalar sequences. It is *Besselian* if there is $A > 0$ such that

$$A \sum_n |a_n|^2 \leq \left\| \sum_n a_n x_n \right\|^2$$

for all finitely supported scalar sequences. These terms are used here in their synthesis-inequality sense; terminology can vary in the literature.

The Hilbertian and Besselian properties describe two complementary forms of stability associated with basis expansions. While the Hilbertian property controls the synthesis of vectors from square-summable scalar sequences, the Besselian property ensures square-summability of the coefficients of convergent expansions. The simultaneous occurrence of these properties leads to a complete characterization of Riesz bases among Schauder bases in Hilbert spaces, as established in the following theorem.

Theorem 9.2. *Let (x_n) be a Schauder basis for a Hilbert space $\mathcal{H}$. Then (x_n) is a Riesz basis for $\mathcal{H}$ if and only if it is both Hilbertian and Besselian.*

Proof. The two inequalities in the definition are precisely the Riesz inequalities. Since a Schauder basis is complete, the Riesz-basis characterization applies directly.

For clarity, these definitions also agree, for Schauder bases, with the corresponding convergence conditions. The upper estimate implies convergence for every $a \in \ell^2$ by the Cauchy criterion. Conversely, if every such series converges, uniform boundedness applied to the finite synthesis maps gives the upper estimate. The lower estimate implies that every convergent expansion has square-summable coefficients, by boundedness of its partial sums. Conversely, suppose every convergent basis expansion has square-summable coefficients. The coordinate map $J : \mathcal{H} \to \ell^2$ is then everywhere defined and has closed graph: convergence in either space implies convergence of each coordinate, and the basis coordinate functionals are continuous. The closed graph theorem gives $\|Jx\|_2 \leq C\|x\|$, which yields the lower estimate on finite sums. □

Example 9.3. For the standard orthonormal basis (e_n) of ℓ^2, put $x_n = e_n/n$ and $z_n = ne_n$. Both are unconditional Schauder bases because nonzero diagonal rescaling preserves the partial-sum projections. For finitely supported a,

$$\left\|\sum a_n x_n\right\|^2 = \sum |a_n|^2/n^2, \qquad \left\|\sum a_n z_n\right\|^2 = \sum n^2|a_n|^2.$$

The first sequence satisfies the upper estimate with constant one but fails every positive lower estimate; the second satisfies the lower estimate with constant one but fails every finite upper estimate. The failures follow by testing single coordinates. Equivalently, $\sum x_n$ converges despite its coefficients not belonging to ℓ^2, whereas $\sum z_n/n$ diverges despite square-summable coefficients. Thus neither basis is Riesz, and the two properties are independent.

10. Frames and bases: similarities and differences

Frames and bases constitute two fundamental approaches to the representation of elements in Hilbert spaces. While bases emphasize uniqueness of expansion, frames allow redundancy without sacrificing completeness or stability of reconstruction. This additional flexibility makes frames particularly useful in approximation theory, signal processing, and numerical analysis. Despite their differences, frames and bases share several important structural and operator-theoretic properties. The present section examines these similarities and differences, with particular emphasis on the relationship between exact frames and Riesz bases. Frame inequalities control analysis coefficients rather than arbitrary synthesis coefficients. The

distinction allows redundancy and leads to an operator-theoretic criterion for exactness [2, 1, 4].

Definition 10.1. A sequence (f_n) is a *frame* for $\mathcal{H}$ if there exist $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_n |\langle x, f_n\rangle|^2 \le B\|x\|^2 \qquad (x \in \mathcal{H}).$$

It is a *Parseval frame* if $A = B = 1$, and an *exact frame* if removing any one vector destroys the frame property.

The essential distinction between a general frame and a Riesz basis lies in the presence or absence of redundancy. From an operator-theoretic perspective, this distinction is reflected in the kernel of the synthesis operator. Although every frame admits a stable reconstruction of elements, the uniqueness of square-summable coefficient representations requires the absence of redundancy. The following theorem establishes the equivalence of these structural, operator-theoretic, and representation properties, thereby characterizing Riesz bases within the class of frames.

Theorem 10.2. *For a frame (f_n) in a Hilbert space, the following are equivalent:*

(i) (f_n) is a Riesz basis;
(ii) (f_n) is exact;
(iii) the synthesis operator $T : \ell^2 \to \mathcal{H}$, $T(a_n) = \sum_n a_n f_n$, is injective;
(iv) the frame has a unique coefficient representation for every vector using ℓ^2 coefficients.

Proof. For a frame, T is bounded and surjective, since its frame operator TT^* is boundedly invertible. If T is injective, the bounded inverse theorem makes it a bounded isomorphism, so the sequence is Riesz. A Riesz basis is exact because deleting one vector leaves a system whose closed span is the image under an isomorphism of a proper closed coordinate subspace. Conversely, if T has a nonzero kernel vector $a = (a_n)$, choose j with $a_j \neq 0$. Then $f_j = -\sum_{n\neq j}(a_n/a_j)f_n$ with an ℓ^2 coefficient sequence. This implies the remaining vectors still form a frame: for any $b \in \ell^2$, subtract $(b_j/a_j)a$ to obtain a coefficient vector with zero jth entry and norm bounded by a constant times $\|b\|$. Since T is onto, the restricted synthesis operator is onto and hence gives a frame. Thus exactness implies injectivity. Finally, injectivity is equivalent to uniqueness of ℓ^2 synthesis coefficients. □

Example 10.3 (A redundant Parseval frame)**.** Let $f_{2n-1} = f_{2n} = e_n/\sqrt{2}$ in ℓ^2. For $x = \sum a_n e_n$, Parseval's identity gives

$$\sum_j |\langle x, f_j\rangle|^2 = \sum_n |a_n|^2 = \|x\|^2, \qquad x = \sum_j \langle x, f_j\rangle f_j.$$

The synthesis operator is

$$Tc = \sum_n \frac{c_{2n-1} + c_{2n}}{\sqrt{2}} e_n, \qquad \ker T = \{(t_1, -t_1, t_2, -t_2, \ldots) : t \in \ell^2\}.$$

In particular, all square-summable representations of x have the form $c_{2n-1} = a_n/\sqrt{2} + t_n$, $c_{2n} = a_n/\sqrt{2} - t_n$, with $t \in \ell^2$. The nonzero finite relation $f_1 - f_2 = 0$ rules out both the Schauder and Riesz basis properties. Deleting one vector leaves a frame with optimal bounds $1/2$ and 1, so the original frame is not exact. Deleting both copies of any e_n destroys completeness. This example separates redundancy from uniqueness without sacrificing stable reconstruction.

Remark 10.4. A *Schauder frame* is a pair (x_n, φ_n) with $x = \sum_n \varphi_n(x)x_n$ for every x. The vectors need not be linearly independent and the coefficients need not be unique. A Schauder basis gives a Schauder frame through its coefficient functionals, but the converse is false. A frame in the usual Hilbert-space sense provides a Schauder-frame expansion via its canonical dual, although it may be redundant.

11. Summary

The preceding discussion reveals a hierarchy of basis systems in Hilbert spaces, distinguished by their geometric, topological, and representation properties. Although several of these notions are closely related, their defining conditions differ substantially in strength and generality. It is therefore useful to consolidate the established results into a unified framework that clarifies the logical implications among various classes of bases and frames. The following summary highlights these interrelationships and identifies the conditions under which certain implications become equivalences. For a complete sequence in a separable Hilbert space, the following implications hold:

$$\boxed{\text{Orthonormal basis}} \Longrightarrow \boxed{\text{Riesz basis}}$$
$$\Longrightarrow \boxed{\text{Unconditional Schauder basis}}$$
$$\Longrightarrow \boxed{\text{Schauder basis}} \Longrightarrow \boxed{\text{Strong } M\text{-basis}}.$$

Each reverse implication can fail. For the last one, enumerate an orthonormal basis after applying finite-dimensional invertible block maps whose within-block projection norms tend to infinity. The resulting biorthogonal pair is strong because each finite block reconstructs exactly from its nonzero coordinates and the orthogonal block sums converge; it is not a Schauder basis in block order. For instance, in the nth two-dimensional block use e_{2n-1} and $e_{2n-1} + n^{-1}e_{2n}$: the projection onto the first vector along the second sends e_{2n} to $-ne_{2n-1}$. The second reverse implication becomes true with the extra assumption of semi-normalization. The first reverse implication fails because invertible operators need not preserve orthogonality. Also,

$$\boxed{\text{normalized unconditional Schauder basis}} \Longleftrightarrow \boxed{\text{normalized Riesz basis}}.$$

Here "normalized Riesz basis" means a Riesz basis whose vectors have unit norm. The class of normalized Schauder bases is strictly larger. An orthogonal Schauder basis is Riesz precisely when semi-normalized. A complete system which is both Hilbertian and Besselian is Riesz. Hamel bases and general frames are not on the same chain of implications.

12. Further observations

The results established in the preceding sections provide a systematic understanding of the structural relationships among various basis systems in Hilbert spaces. Nevertheless, several additional aspects deserve attention, particularly those concerning the limitations of the established implications and the role of supplementary assumptions. The following observations offer further insight into the behavior of these systems and highlight subtle distinctions that are relevant to the broader study of bases and frames.

1. **Stability.** If (x_n) is a Riesz basis with lower bound A and (z_n) satisfies
$$\left\| \sum_n a_n(x_n - z_n) \right\| \le \delta \left(\sum_n |a_n|^2 \right)^{1/2} \quad \text{for all finitely supported } (a_n),$$
where $0 \le \delta < \sqrt{A}$, then (z_n) is a Riesz basis. Indeed, the synthesis operators differ in norm by at most δ, and a Neumann-series argument applied to $T_x^{-1}T_z$ proves invertibility. The new Riesz bounds can be taken as $(\sqrt{A} - \delta)^2$ and $(\sqrt{B} + \delta)^2$.
2. **Ordering.** The Schauder-basis property can depend on the order of the vectors. Unconditional bases retain the basis property under every permutation; a conditional basis has at least one permutation that is not a Schauder basis.
3. **Normalization.** Scaling by nonzero coefficients preserves Schauder expansions and unconditionality but may destroy Riesz bounds. Uniformly bounded scaling above and away from zero preserves the Riesz property.
4. **Coefficient spaces.** A Riesz basis has coefficient space ℓ^2 with equivalent norms. For a general Schauder basis, its natural coefficient space may be a different Banach sequence space.
5. **Frames versus bases.** Some redundant frames permit stable reconstruction after deletion of selected frame vectors, while a Riesz basis provides uniqueness. This distinction motivates frame constructions in sampling and signal analysis [3].

13. Conclusion

In infinite-dimensional Hilbert spaces, completeness, minimality, unique norm-convergent expansion, unconditionality, and two-sided ℓ^2 estimates must be distinguished. Riesz bases are exactly the complete systems that are equivalent to orthonormal bases, and exactly the semi-normalized unconditional Schauder bases. The weighted Fourier example shows decisively that even a normalized Schauder basis need not be Riesz. Biorthogonal and Markushevich systems clarify uniqueness at the coordinate level, whereas frames allow redundancy. These distinctions are foundational in basis theory, operator theory, and frame-based approximation.

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